# A SIMPLE COVARIANCE-BASED CHARACTERIZATION OF JOINT SIGNAL REPRESENTATIONS OF ARBITRARY VARIABLES


*Akbar M. Sayeed* *Douglas L. Jones*

Coordinated Science Laboratory*
University of Illinois
Urbana, IL 61801
akbar@csl.uiuc.edu jones@csl.uiuc.edu



## ABSTRACT

Joint signal representations of arbitrary variables extend the scope of joint time-frequency representations, and provide a useful description for a wide variety of nonstationary signal characteristics. Cohen's marginal-based theory for bilinear representations is canonical from a distributional viewpoint, whereas, from other perspectives, such as characterization of the effect of unitary signal transformations of interest, a covariance-based formulation is needed and more attractive. In this paper, we present a simple covariance-based characterization of bilinear joint signal representations of arbitrary variables. The formulation is highlighted by its simple structure and interpretation, and naturally extends the concept of the corresponding linear representations.


## 1. INTRODUCTION

Joint signal representations in terms of physical quantities other than time and frequency have recently been investigated by a number of authors [1, 2, 3, 4, 5, 6]. For example, joint time-scale representations (TSRs) [1, 2] analyze signal characteristics in terms of time and scale content. The motivation for studying such generalized joint representations is to develop tools that can provide a useful description for a broad class of nonstationary signal characteristics.

In existing literature, the construction of joint signal representations (JSRs) has been based on two main approaches. Cohen's pioneering method for constructing bilinear time-frequency representations (TFRs) interprets the TFRs as quasi-energy distributions which satisfy certain marginal constraints analogous to probability distributions. The other main approach is to consider arbitrary quadratic forms in the signal, parameterized by variables of interest, and then to impose certain *covariance* constraints to characterize a certain class of JSRs. The concept of covariance relates prescribed changes in the signal to corresponding changes in the JSR, in a well-defined and consistent manner. For example, the affine class of TSRs characterizes the representations which are covariant to time-shifts and scalings; that is, the TSRs undergo similar changes in the variables (time and scale) [1, 2]. Mathematically, such signal changes are characterized by parameterized unitary transformations. Covariance properties are particularly important in situtions in which certain changes in the signal (changes in the parameters of the unitary operator) need to be estimated or detected. For example, the use of TFRs and TSRs for optimal detection of signals with unknown time, frequency or scale changes, crucially depends on such properties [7, 8].

In view of the recent interest in generalized JSRs, Cohen has recently extended his original marginal-based method to joint representations of *arbitrary* variables [3, 9]. A similar approach was proposed by Baraniuk [10] and was shown by Sayeed and Jones to be equivalent to Cohen's [11]. Cohen's marginal-based approach seems fairly complete, and is canonical from a distributional viewpoint since the representations measure the distribution of signal energy as a function of the variables. Some results on a covariance-based generalization for two variables have been recently reported by Hlawatsch and Bölcskei [12, 13]. However, as we will elaborate later, their approach is restricted to the case of two variables and has a rather complicated formulation that does not highlight some of the key features of the covariance-based approach.

In this paper, we present a simple covariance-based characterization of JSRs of arbitrary variables, and our formulation yields a canonical method for generating such JSRs. Our theory, when restricted to the case of two variables, is equivalent to that proposed in [12, 13] but, as we will see, it has a much simpler, direct form that makes it conceptually more attractive as well.

## 2. PRELIMINARIES

We assume that the signals of interest belong to a closed subspace $\mathcal{H}$ of the Hilbert space $L^2(\mathbb{R})$ of finite-energy signals. Let $G \subset \mathbb{R}^N$ be a parameter set and let $\{\mathbf{U}_g\}_{g \in G}$ be a family of unitary operators defined on $\mathcal{H}$; that is, for any $g \in G$, $\mathbf{U}_g : \mathcal{H} \to \mathcal{H}$ and $\langle \mathbf{U}_g s, \mathbf{U}_g s \rangle = \langle s, s \rangle$ for all $s \in \mathcal{H}$ where $\langle \cdot, \cdot \rangle$ denotes the inner product defined on $L^2(\mathbb{R})$. For a given $g = (g_1, g_2, \cdots, g_N) \in G$, each "coordinate", $g_i$, represents a variable or physical quantity of interest.

The family of unitary operators $\{\mathbf{U}_g\}$ represents signal transformations that are of interest to us; for example, the time-frequency shift operator or the time-scale shift operator (time-shifts and scalings) for $N = 2$ (see the examples in Section 4). In many cases, certain natural constraints on the operators $\{\mathbf{U}_g\}$ dictate that $G$ is a group (we denote group operation by $\bullet$) and $\mathbf{U}_g$ is unitary representation of


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$G$ on $\mathcal{H}$ [2, 14, 15]; that is,

$$\mathbf{U}_a \mathbf{U}_b = \mathbf{U}_{a \bullet b} \quad \text{(within a phase factor}^1\text{)} \tag{1}$$

and this will be assumed throughout the paper.

Although we have considered arbitrary group representations of the form $\{\mathbf{U}_g\}$, it is worth noting that in most cases of interest, the operator $\mathbf{U}_g$ will be a composition of $N$ unitary operators that are themselves unitary representations of one-parameter groups.[2] The reason is that, usually, individual variables of interest, such as time, frequency or scale, are associated with operators and then the joint representations are constructed.

## 3. A CANONICAL COVARIANCE-BASED CHARACTERIZATION

Suppose that $\{\mathbf{U}_g\}$ is a group of unitary operators and we are interested in bilinear (quadratic) JSRs that react in a covariant fashion to the transformation of signals by $\mathbf{U}_g$. For example, in [1], the affine class is defined as the class of quadratic JSRs that are covariant to the time-scale shift operator (see Section 4.2). A natural notion of covariance is provided by the group relation; that is, the JSRs should react to the unitary transformation $\mathbf{U}_g$ as

$$(\mathbf{P}\mathbf{U}_g s)(a) = (\mathbf{P}s)(g^{-1} \bullet a) \quad \text{for all} \quad a, g \in G\ , \tag{2}$$

where the signal $s$ belongs to $\mathcal{H}$ and the quadratic JSR is denoted by the operator $\mathbf{P}$ which maps the signal into the space of (possibly) complex-valued functions defined on $G$.[3] Recall that each "coordinate", $a_i$, of an element $a$ of $G$ represents a variable or quantity of interest. The following theorem provides a simple characterization of all bilinear JSRs covariant to $\mathbf{U}_g$.

**Theorem.** For any bilinear JSR $\mathbf{P}$ satisfying (2), there exists a linear operator $\mathbf{K}_\mathbf{P} : \mathcal{H} \to \mathcal{H}$ such that for all $s \in \mathcal{H}$

$$(\mathbf{P}s)(a) \equiv \langle \mathbf{K}_\mathbf{P} \mathbf{U}_{a^{-1}} s, \mathbf{U}_{a^{-1}} s \rangle\ ,\ a \in G\ . \tag{3}$$

Conversely, any linear operator $\mathbf{K}_\mathbf{P} : \mathcal{H} \to \mathcal{H}$ defines a bilinear JSR via (3) which satisfies the covariance relation (2).

**Proof.** First suppose that $\mathbf{P}$ is defined by (3). Then, we have

$$\begin{aligned} (\mathbf{P}\mathbf{U}_g s)(a) &= \langle \mathbf{K}_\mathbf{P} \mathbf{U}_{a^{-1}} \mathbf{U}_g s, \mathbf{U}_{a^{-1}} \mathbf{U}_g s \rangle \\ &= \left\langle \mathbf{K}_\mathbf{P} \mathbf{U}_{(g^{-1}\bullet a)^{-1}} s, \mathbf{U}_{(g^{-1}\bullet a)^{-1}} s \right\rangle \\ &= (\mathbf{P}s)(g^{-1} \bullet a) \end{aligned} \tag{4}$$

and thus $\mathbf{P}$ satisfies (2). Conversely, suppose that $\mathbf{P}$ is an arbitrary bilinear representation that satisfies (2). It follows that there exists a family of linear operators $\{\mathbf{K}_a\}$ such that $(\mathbf{P}s)(a) = \langle \mathbf{K}_a s, s \rangle$, $a \in G$. It follows from (2) that

$$\langle \mathbf{K}_a \mathbf{U}_g s, \mathbf{U}_g s \rangle = \langle \mathbf{K}_{g^{-1}\bullet a} s, s \rangle \tag{5}$$

for all $s \in \mathcal{H}$ and for all $a, g \in G$. By setting $a = \theta$ and substituting $g$ for $g^{-1}$, (5) yields

$$(\mathbf{P}s)(g) = \langle \mathbf{K}_g s, s \rangle = \langle \mathbf{K}_\theta \mathbf{U}_{g^{-1}} s, \mathbf{U}_{g^{-1}} s \rangle \tag{6}$$

which completes the proof.[4]

Thus, the covariance properties of the representations are determined by $\mathbf{U}_a$ in (3), and all other properties are completely determined by the linear operator $\mathbf{K}_\mathbf{P}$.[5] The choice of the operator $\mathbf{K}_\mathbf{P}$ in controlling the properties of the representation is completely equivalent to the choice of the kernel in Cohen's and affine classes.

### 3.1. Relationship to the approach in [12,13]

As mentioned earlier, in [12, 13], Hlawatsch and Bölcskei propose a covariance-based generalized theory for the case of two variables (N=2). In particular, they consider generalized *time-frequency* representations, which necessarily requires a remapping of coordinates (into time and frequency) that turns out to have a complicated form in their construction. Our formulation facilitates such a remapping in a very simple and direct manner. Essentially, the remapping is accomplished with a one-to-one and onto mapping $\psi : \mathbb{R}^2 \to G$ between the coordinates[6] [12, 13, 5], and, thus, a remapped generalized *time-frequency* representation, $\widehat{\mathbf{P}}$, is obtained as ($b \equiv (t, f) \in \mathbb{R}^2$)

$$\begin{aligned} (\widehat{\mathbf{P}}s)(t,f) \equiv (\widehat{\mathbf{P}}s)(b) &\equiv (\mathbf{P}s)(\psi(b)) \\ &= \left\langle \mathbf{K}_\mathbf{P} \mathbf{U}_{(\psi(b))^{-1}} s, \mathbf{U}_{(\psi(b))^{-1}} s \right\rangle . \end{aligned} \tag{7}$$

However, we emphasize that in many situations, such as those involving signal detection via JSRs [7, 8, 17], such a remapping of coordinates is unnecessary.

### 3.2. Interpretation in terms of linear JSRs

Further insight into the interpretation of (3) can be gained by using the singular value decomposition of the operator $\mathbf{K}_\mathbf{P}$ (if it is compact):[7]

$$\begin{aligned} (\mathbf{P}s)(a) &= \sum_k \sigma_k \langle \mathbf{U}_{a^{-1}} s, v_k \rangle \langle u_k, \mathbf{U}_{a^{-1}} s \rangle \\ &= \sum_k \sigma_k \langle s, \mathbf{U}_a v_k \rangle \langle \mathbf{U}_a u_k, s \rangle \end{aligned} \tag{8}$$

which implies that the value of $\mathbf{P}s$ at a particular value of $a$ is completely determined by the projection of $\mathbf{U}_{a^{-1}} s$ onto

[1] In certain cases $\mathbf{U}_a \mathbf{U}_b = e^{j\psi(a,b)} \mathbf{U}_{a\bullet b}$ but since we are mainly interested in quadratic signal representations, this phase factor will not be an issue. Even for linear representations such a phase factor is inconsequential in most cases.

[2] More precisely, the underlying group $G$ is a Lie group [14].

[3] Related group theoretic and covariance-based arguments (coadjoint representations and the method of orbits [14]) are used in [2] to derive analogues of the Wigner distribution for the affine group, and in [16] to define wideband ambiguity functions.

[4] We note that in [2], the Bertrands arrive at a specialized version of (6) for JSRs covariant to time-shifts and scalings.

[5] Operators of the form $\mathbf{U}_g \mathbf{K} \mathbf{U}_g^{-1}$ are also discussed in [16] with regard to covariance properties of phase space functions defined on $G$.

[6] Note that $(t, f) \in \mathbb{R}^2$ and $G \subset \mathbb{R}^2$ in this case.

[7] Similar expansions for Cohen's class of TFRs are discussed in [18] and [19].

the singular vectors, $u_k$'s and $v_k$'s, and the singular values $\sigma_k$'s. If $\mathbf{K_P}$ is Hermitian, then

$$(\mathbf{P}s)(a) = \sum_k \lambda_k |\langle s, \mathbf{U}_a u_k \rangle|^2 \tag{9}$$

where the $\lambda_k$'s are the eigenvalues and the $u_k$'s are the eigenfunctions. In particular, (9) implies that the resulting representation is real-valued. Moreover, if $\mathbf{K_P}$ is rank-1 then (9) reduces to $(\mathbf{P}s)(a) = \lambda|\langle s, \mathbf{U}_a u\rangle|^2$. The linear transform

$$(\mathbf{L}s)(a;u) \equiv \langle s, \mathbf{U}_a u\rangle = \int s(t)(\mathbf{U}_a u)^*(t)dt \tag{10}$$

is a generalization of the short-time Fourier transform or the wavelet transform, where $u$ is analogous to the analyzing window or wavelet. Thus, from (9) we note that any arbitrary (real-valued) bilinear JSR (corresponding to a compact operator) can be thought of as a weighted sum of the squared-magnitudes of a bank of linear JSRs $(\mathbf{L}s)(a;u_k) = \langle s, \mathbf{U}_a u_k\rangle$.

### 3.3. Equivalence classes of JSRs

Two groups $G$ and $G'$ are isomorphic if there exists a one-to-one and onto mapping[8] $\Psi : G' \to G$. For a given $N$, the space of $N$-parameter groups can be partitioned into *equivalence classes* of isomorphic groups. Given a group $G$ and a unitary representation $\mathbf{U}_g$ of $G$ on $\mathcal{H}$, (3) characterizes the class, $\mathcal{C}$, of bilinear JSRs covariant to $\mathbf{U}_g$. However, there are infinitely many unitary representations of $G$ on $\mathcal{H}$ and all of them can be generated from $\mathbf{U}_g$ by using different unitary transforms $\mathbf{W} : \mathcal{H} \to \mathcal{H}$ as

$$\mathbf{U}'_g = \mathbf{W}^{-1}\mathbf{U}_g\mathbf{W} \ . \tag{11}$$

For a given $\mathbf{W}$ and the associated unitary representation, $\mathbf{U}'_g$, it follows from (3) that the corresponding class, $\mathcal{C}'$, of bilinear JSRs can be generated from $\mathcal{C}$ as

$$\mathcal{C}' = \{\mathbf{PW} : \mathbf{P} \in \mathcal{C}\} \tag{12}$$

making $\mathcal{C}'$ and $\mathcal{C}$ unitarily equivalent [5, 20, 11].

Similarly, for any group $G'$ from the equivalence class of $G$, with the isomorphism given by $\Psi : G' \to G$, a unitary representation of $G'$ on $\mathcal{H}$ can be generated from $\mathbf{U}_g$, by [14, 11]

$$\mathbf{U}''_{g'} = \mathbf{U}_{\Psi(g')} \ . \tag{13}$$

Thus, the class $\mathcal{C}''$ of JSRs covariant to $\mathbf{U}''_{g'}$ can be generated from $\mathcal{C}$ via an axis-warping transformation [5, 4, 11] as

$$\mathcal{C}'' = \{\mathbf{VP} : (\mathbf{VP}s)(g') = (\mathbf{P}s)(\Psi(g')), \ \mathbf{P} \in \mathcal{C}\} \ . \tag{14}$$

We note that the remapping of coordinates into time-frequency, as in (7), can also be thought of as transforming the JSR (based on a group $G$) into another one corresponding to another group that is isomorphic to $G$. It can be shown that such a remapping of coordinates is an isometry [5, 11, 13]. Thus, we conclude that for a given $G$, and a unitary representation $\mathbf{U}_g$ of $G$, the *classes* of JSRs corresponding to *all* the groups from the equivalence class of $G$, and *all* the corresponding unitary representations, can be generated from the class $\mathcal{C}$ (defined by $\mathbf{U}_g$) via

$$\mathcal{C}''' = \{\mathbf{VPW} : (\mathbf{VPW}s)(g''') = (\mathbf{PW}s)(\Psi(g''')), \ \mathbf{P} \in \mathcal{C}\} \tag{15}$$

by choosing different unitary transforms $\mathbf{W} : \mathcal{H} \to \mathcal{H}$ and isomorphisms $\Psi : G' \to G$ between $G$ and another group $G'$ in the equivalence class. Hence, all the classes of JSRs corresponding to a particular equivalence class of groups are unitarily equivalent to each other [5, 20, 11].

## 4. EXAMPLES

We now illustrate our method by applying the Theorem to some well-known classes JSRs.

### 4.1. Cohen's class of bilinear TFRs.

Let $\mathcal{H} = L^2(\mathbb{R})$ and let $G = \mathbb{R}^2$ with the group operation defined by $(x_1, y_1) \bullet (x_2, y_2) = (x_1 + y_1, x_2 + y_2)$;[9] $(x, y)^{-1} = (-x, -y)$. For $\tau$, $\nu \in \mathbb{R}$, define the time-shift and frequency-shift operators as $(\mathbf{T}_\tau s)(x) = s(x - \tau)$ and $(\mathbf{F}_\nu s)(x) = e^{j2\pi\nu x}s(x)$, respectively. Let $a = (t, f)$ and define the time-frequency-shift operator as $\mathbf{U}_{(t,f)} = \mathbf{F}_\nu \mathbf{T}_\tau$, which satisfies the group composition law (1). Using (3) in the Theorem, the class of JSRs covariant to time-frequency shifts is characterized by

$$\begin{aligned}
(\mathbf{P}s)(t,f) &= \langle \mathbf{K_P}\mathbf{U}_{(t,f)^{-1}}s, \mathbf{U}_{(t,f)^{-1}}s\rangle \\
&= \langle \mathbf{K_P}\mathbf{F}_{-f}\mathbf{T}_{-t}s, \mathbf{F}_{-f}\mathbf{T}_{-t}s\rangle \\
&= \int\int K_{\mathbf{P}}(u_2, u_1)s(u_1 + t)s^*(u_2 + t) \\
&\quad e^{-j2\pi f(u_1 - u_2)}du_1 du_2 \\
&= \int\int \Phi(u, \tau)s(t + u + \tau/2) \\
&\quad s^*(t + u - \tau/2)e^{-j2\pi f\tau}du d\tau
\end{aligned} \tag{16}$$

where $K_{\mathbf{P}}$ is the kernel corresponding to the operator $\mathbf{K_P}$, and $\Phi(u, \tau) = K_{\mathbf{P}}(u - \tau/2, u + \tau/2)$. We note that (16) is a familiar expression for Cohen's class [3], and that this operator-based characterization of Cohen's class is also used in [18].

### 4.2. Affine class of bilinear TSRs.

Let $\mathcal{H} = L^2(\mathbb{R})$ and $G = \mathbb{R}\times(0, \infty)$, and let the group operation be defined by $(t_1, c_1)\bullet(t_2, c_2) = (t_1 + c_1 t_2, c_1 c_2)$ (affine group); $(t, c)^{-1} = (-t/c, 1/c)$. Define the dilation operator as $(\mathbf{D}_c s)(x) = \frac{1}{\sqrt{c}}s(x/c)$ and the time-scale shift operator as $\mathbf{U}_{(t,c)} = \mathbf{T}_t\mathbf{D}_c$, which satisfies the composition group law (1). Using (3) in the Theorem, the class of bilinear JSRs covariant to time-shifts and scalings is characterized by

$$\begin{aligned}
(\mathbf{P}s)(t,c) &= \langle \mathbf{K_P}\mathbf{U}_{(t,c)^{-1}}s, \mathbf{U}_{(t,c)^{-1}}s\rangle \\
&= \langle \mathbf{K_P}\mathbf{T}_{-t/c}\mathbf{D}_{1/c}s, \mathbf{T}_{-t/c}\mathbf{D}_{1/c}s\rangle \\
&= c\int\int\int K_{\mathbf{P}}(u_2, u_1)W_s(c(u_1 + u_2)/2 + t, f) \\
&\quad e^{j2\pi fc(u_1 - u_2)}df du_1 du_2
\end{aligned}$$

[8] That is continuous and has a continuous inverse [14, 11].

[9] Can be thought of as a subgroup of the Weyl-Heisenberg group [14].

$$= \iint \Pi((u-t)/c, fc) W_s(u,f) du df \qquad (17)$$

where $W_s$ is the Wigner distribution of $s$ defined by $W_s(t,f) = \int s(t+\tau/2)s^*(t-\tau/2)e^{-j2\pi f\tau}d\tau$ and the kernel $\Pi$ is related to $K_{\mathbf{P}}$ as $\Pi(u,f) = \int K_{\mathbf{P}}(u+\tau/2, u-\tau/2)e^{-j2\pi f\tau}d\tau$. Note that (17) is a familiar characterization of the affine class [1].

Similarly, other covariance-based classes, such as the hyperbolic class covariant to scalings and hyperbolic time-shifts [4], and the power class covariant to scalings and power time-shifts [6], can be characterized by using the remapped characterization in (7).

## 5. CONCLUSION

Cohen's generalized method for constructing JSRs of arbitrary variables, although canonical from a distributional viewpoint, is not adequate, in general, for characterizing the effect of unitary transformations on signals; a covariance-based approach is needed in such situations (for example, in signal detection scenarios). In this paper, we have presented a simple characterization of JSRs having arbitrary group covariance properties with respect to given unitary signal transformations based on the group. Our formulation yields a canonical method for constructing such signal representations, and provides a simple interpretation in terms of corresponding linear JSRs.

Our method is valid for an arbitrary number of variables, and, in the case of two variables, is equivalent to the approach presented in [12, 13]. However, our formulation is much simpler and more direct, making it conceptually more attractive. In particular, in [12, 13] generalized *time-frequency* representations are considered which necessarily involves a remapping of coordinates that turns out to be rather unwieldy and complicated in their method. Our characterization, on the other hand, facilitates such a remapping in a very simple and transparent manner.

Finally, we mention that in the marginal-based approach of Cohen's, the covariance properties are difficult to analyze in general, and in the covariance-based method, the marginal properties become nontrivial to characterize. Some preliminary results on such issues have been recently reported [21, 13] but still more work needs to be done to completely bridge the gap between the methodologies of the two approaches. The results presented in this paper should facilitate bridging the gap.

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