Gravitational Forces in Dual-Porosity Systems
I. Model Derivation by Homogenization

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GRAVITATIONAL FORCES IN DUAL-POROSITY SYSTEMS I. MODEL DERIVATION BY HOMOGENIZATION*

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Abstract. We consider the problem of modeling flow through naturally fractured porous media. In this type of media, various physical phenomena occur on disparate length scales, so it is difficult to properly average their effects. In particular, gravitational forces pose special problems. In this paper we develop a general understanding of how to incorporate gravitational forces into the dual-porosity concept. We accomplish this through the mathematical technique of formal two-scale homogenization. This technique enables us to average the single-porosity, Darcy equations that govern the flow on the finest (fracture thickness) scale. The resulting homogenized equations are of dual-porosity type. We consider three flow situations, the flow of a single component in a single phase, the flow of two fluid components in two completely immiscible phases, and the completely miscible flow of two components.

1. Introduction.

It has long been known that fracture systems play a significant role in the flow of fluids in porous media. We consider a naturally fractured porous medium that has throughout its extent a system of interconnected fracture planes, which we idealize as a periodic medium as shown in Fig. 1. If we stay above the Darcy (pore) scale, then three scales exist in this system, the scale of the fracture thickness, the scale of the average distance between fracture planes, and the scale of the entire porous medium. The latter scale is very large, so tractable computer simulation requires that effects on the two finer scales be averaged.

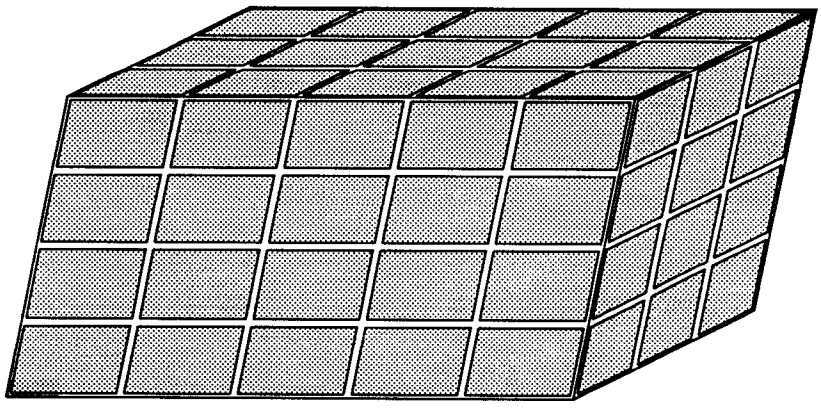

Fig. 1. The idealized periodic medium.

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The difficulty is in averaging the system in such a way that effects on the intermediate scale are not completely lost. Most of the fluid resides in the porous rock (or *matrix*), but most of the flow takes place in the high permeability fractures. The critical physical process to model is the transfer of fluids between the matrix and fracture systems. This transfer takes place on the intermediate scale.

The macroscopic concept of dual-porosity is used to model the flow on its various scales [20, 8, 25]. Briefly, one views the fracture system as a porous structure distinct from the porous structure of the matrix rock itself. Fluid flows macroscopically in the fracture system on the largest scale only, and it is influenced by the intermediate scale matrix flow through a macroscopically distributed source or sink, i.e., a "transfer function." It is generally assumed that no direct flow between matrix blocks occurs; that is, all flow between matrix blocks must go through the fracture system. Flow in each matrix block is calculated on the intermediate scale. It is influenced by the fracture flow through the imposition of a boundary condition. The difficulty in properly formulating this type of model is that one must relate fracture quantities defined on the largest scale to matrix quantities defined on the intermediate scale.

Compressional, diffusive, capillary, gravitational, and viscous forces are important in moving fluids between the matrix and fracture systems. On the basis of physical arguments, many authors have been successful in including the effects of the first three forces in dual-porosity models. To a lesser extent, and then only in certain cases, they are successful in modeling gravitational and viscous forces, but a general way to model these forces in a dual-porosity context seems to be lacking. See, for example, the papers [2, 3, 8, 12, 13, 17, 18, 19, 22, 23, 24, 25], and the many references therein.

More recently, dual-porosity models have been derived through an averaging technique that is sometimes called *formal two-scale mathematical homogenization* [7, 6, 11]. The advantage of this technique is that the macroscopic, dual-porosity model is derived by averaging an accepted, microscopic model of the flow. General descriptions of homogenization can be found in [9, 21, 14].

The homogenized "tall-block" model of [7] properly incorporates gravitational effects in a very special way, because the microscopic equations are averaged only in the two nonvertical directions. For full three-dimensional averaging, the homogenization as given in the literature does not properly handle gravitational forces. Either the effects of gravity are omitted from the matrix flow, or a physically inconsistent model is derived. In the latter case, when the fracture system is in gravitational equilibrium, the matrix system is not. This effect will be seen in Section 3. In this paper, we derive dual-porosity models by two-scale homogenization that properly account for the effects of gravity in the matrix. In the process, we develop a general understanding of how gravity is to be represented in the dual-porosity context.

The outline of the paper is as follows. In the next section, we describe in general terms how the two-scale homogenization technique works, and we define some important notation. Sections 3–6 are concerned with three basic flow situations. Section 3 deals with the simplest case of a single component of fluid in a single

phase. The microscopic and macroscopic models are presented, along with a description of some previous macroscopic models. Formal homogenization from the microscopic model to the macroscopic one is carried out in detail in Section 4. Sections 5–6 deal with two components of fluid. In Section 5 the fluids are assumed to separate into two distinct, immiscible phases, while in Section 6 the fluids form a completely miscible single phase.

In Part II of this work [5], it is shown by means of some computational results that the homogenized dual-porosity model for single phase flow derived here approximates very well the microscopic description of the flow.

2. Homogenization as a Geometric Averaging Technique.

In the two-scale homogenization technique considered in this paper, we begin by posing the correct microscopic equations of flow on the porous medium. We then define a homogenization parameter ϵ; $\epsilon = 1$ refers to the given, physical medium. For each $\epsilon < 1$ in a decreasing sequence tending to zero, we construct an "equivalent" porous medium with matrix blocks that are ϵ times the original size in any linear direction (see Fig. 2); ϵ represents the intermediate, fracture spacing scale. On this ϵ-medium, we pose a microscopic model of flow that is in some sense equivalent to the original $\epsilon = 1$ microscopic model. As ϵ tends to zero, the flow on the family of ϵ-media is averaged, and the limiting result is our macroscopic model of the flow.

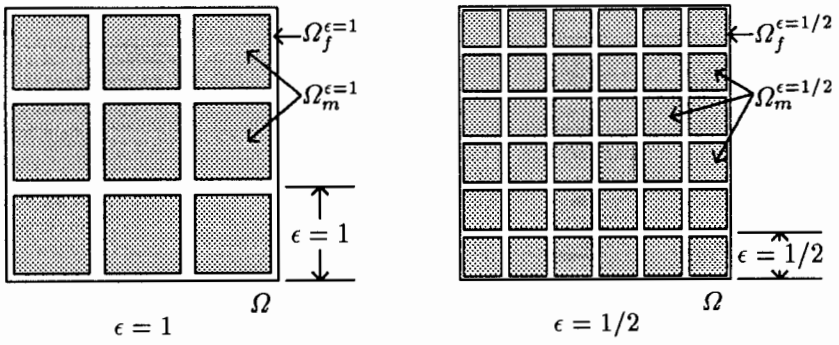

Fig. 2. The first two members of a family of "equivalent" porous media.

We now describe the geometry of our family of porous media. As depicted in Fig. 2, let Ω denote the entire medium, and let Ω_f^ϵ and Ω_m^ϵ denote the fracture and matrix part of the ϵ-medium, respectively. The fractures give each ϵ-medium a periodic structure. Each period is congruent to ϵ times the unit reference cell $\mathcal{Q}$. The fracture part $\mathcal{Q}_f$ completely surrounds the matrix part $\mathcal{Q}_m$ (see Fig. 3). For simplicity, assume that the centroid of $\mathcal{Q}$ is the origin and that $\mathcal{Q}_f$ is connected. Within the ϵ-medium, the period at point $x \in \Omega$ is denoted by $\mathcal{Q}^\epsilon(x)$.

Let e_j denote the standard unit vector in the jth Cartesian direction, where e_3 points in the direction of gravity. In general, x will denote a position in Ω, and y will denote a position in $\mathcal{Q}$ as measured from its centroid. For any $x \in \Omega$, let $x_c^\epsilon(x)$ denote the centroid of the ϵ-period $\mathcal{Q}^\epsilon(x)$ containing x. As depicted in Fig. 3, we define y by $x = x_c^\epsilon(x) + \epsilon y$. For heuristic purposes we may consider that x selects an ϵ-period $\mathcal{Q}^\epsilon(x)$ of Ω on the intermediate scale, while ϵy determines a point within

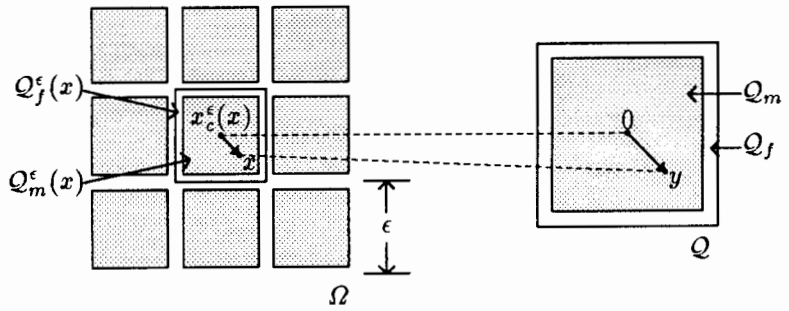

Fig. 3. The unit cell $\mathcal{Q}$ and its relation to a point $x = x_c^\epsilon(x) + \epsilon y$ in the ϵ-medium.

that period on the fine scale; that is, we consider x as $x_c^\epsilon(x)$, and we determine points in the enlarged, congruent period $\mathcal{Q}$ by $y = \epsilon^{-1}(x - x_c^\epsilon(x))$.

In this paper, homogenization (i.e., $\epsilon \to 0$) will result in a macroscopic, dual-porosity model that is formulated in six space dimensions. Three of the space dimensions represent the entire medium Ω over which the fracture system flow occurs. This flow is an average of the flow over the two finer scales, so it is no longer restricted to the physical fractures. At each point $x \in \Omega$, there exists a three dimensional, now infinitely small, matrix block $\mathcal{Q}^0(x)$ congruent to the original cell $\mathcal{Q}$ on which matrix flow occurs on the intermediate scale.

Let ∂ denote "boundary of," and assume that the ϵ's are chosen so that $\partial\Omega \subset \partial\Omega_f^\epsilon$. The matrix-fracture interface is then $\partial\Omega_m^\epsilon$. Let ν denote the outward unit normal vector to this surface ($\partial\Omega_m^\epsilon$ or $\partial\mathcal{Q}_m$).

Obviously, the success of our homogenization is critically dependent on whether the ϵ-medium problems are indeed "equivalent" to the $\epsilon = 1$ case. This question can be laid to rest only by field, laboratory, or numerical experiments; numerical results are presented in Part II of this work [5]. However, we can expect to obtain a reasonable macroscopic model of the flow only if the microscopic description satisfies the following four properties.

(P1) The correct microscopic model of Darcy flow is obtained if $\epsilon = 1$.

(P2) For each ϵ, Darcy flow under the influence of gravity occurs in the fractures and within the *scaled* matrix blocks; that is, if any matrix block $\mathcal{Q}_m^\epsilon$ is expanded to unit size $\mathcal{Q}_m$, the transformed equations reflect Darcy's law.

(P3) If the fracture system is in equilibrium in the vicinity of a matrix block, that block's boundary condition reflects this equilibrium.

(P4) If the entire system is in equilibrium in the vicinity of a matrix block, there must be agreement between the total mass of each phase in that matrix block as calculated from the scaled and unscaled versions of the governing equations.

Property (P1) assumes that we stay above the Darcy scale, so that Darcy's law governs the system when $\epsilon = 1$. Property (P2) is necessary to describe the physics of the flow for $\epsilon < 1$. Properties (P3) and (P4) are new. The effect of (P3) is that as ϵ tends to zero, the matrix is influenced by the fracture system in such a way that

the fracture system appears to be in gravitational equilibrium on the intermediate, fracture spacing scale. We include (P4) as a statement of conservation of mass. When we scale the matrix problem as required by (P2)–(P3), we generate changes in the phase pressures which may change the total mass of each phase.

Each of the microscopic models that we consider below will satisfy these four properties for each ϵ, and consequently also for the macroscopic model. In all cases we ignore outer boundary conditions on $\partial\Omega$ since we are interested in internal flow; likewise, we ignore external sources/sinks. We also neglect to specify the initial conditions. For medium and fluid properties, we use subscript f for fracture quantities and m for matrix quantities.

We close this section with some general notation for the medium. Let g denote the gravitational constant. Let ϕ_f^* and k_f^* denote the fracture porosity and absolute permeability defined on the finest, fracture thickness scale (so $\phi_f^* \approx 1$ and k_f^* is very large). Denote by ϕ_m the matrix porosity, and by k_m the matrix permeability. These are defined on the pore scale. Finally, let ϕ_f and k_f denote the fracture *system* quantities defined on the intermediate, fracture spacing scale. These quantities must be determined from homogenization, though clearly

$$\phi_f = \frac{|\mathcal{Q}_f|}{|\mathcal{Q}|}\phi_f^* \tag{2.1}$$

(where $|\cdot|$ denotes the volume or area of the given domain). In general, porosities are pressure dependent and permeabilities are symmetric, positive definite tensors. Apart from the medium's fracture structure, we tacitly assume for convenience that the medium is homogeneous. This assumption is unnecessary [7, 6].

3. Single Phase, Single Component Flow.

In this section, we consider a single component of fluid in a single phase. Let p, $\rho(p)$, and μ denote the fluid pressure, density, and viscosity, respectively. The simplest way to satisfy Property (P3) is to treat pressure gradient effects on the same footing as gravitational effects. This is easily done if the concept of pseudo-potential [16] applies to the system. It is defined by

$$\Phi = \int_{p_0}^{p} \frac{d\pi}{\rho(\pi)g} - x_3, \tag{3.1}$$

where p_0 is some reference pressure and x_3 is the depth. Then $\nabla p - \rho(p)ge_3 = \rho(p)g\nabla\Phi$; i.e., the pseudo-potential determines the flow. Note that

$$p = \psi(\Phi + x_3) \tag{3.2}$$

for the function ψ equal to the inverse of the integral in (3.1) as a function of p; also, ψ satisfies $\psi' = \rho(\psi)g$ and gives the gravitational equilibrium pressure distribution $p = \psi(\bar{\Phi} + x_3)$ for some constant $\bar{\Phi}$.

We now describe our microscopic model defined on the finest scale. It represents Darcy flow imposed over the entire medium on the scale of the fracture thickness,

with porosity and permeability discontinuous across the interface $\partial\Omega_m^\epsilon$. For notational convenience, let $\alpha(p) = (\rho(p))^2 g/\mu$. The fracture flow on the ϵ-medium is governed by

$$\frac{\partial}{\partial t}\left[\phi_f^*(p_f^\epsilon)\rho(p_f^\epsilon)\right] - \nabla\cdot\left[\alpha(p_f^\epsilon)k_f^*\nabla\Phi_f^\epsilon\right] = 0, \quad x\in\Omega_f^\epsilon, \tag{3.3a}$$

$$p_f^\epsilon = \psi(\Phi_f^\epsilon + x_3), \quad x\in\Omega_f^\epsilon, \tag{3.3b}$$

$$\alpha(p_f^\epsilon)k_f^*\nabla\Phi_f^\epsilon\cdot\nu = \epsilon^2\alpha(p_m^\epsilon)k_m\nabla\Phi_m^\epsilon\cdot\nu, \quad x\in\partial\Omega_m^\epsilon. \tag{3.3c}$$

For the matrix,

$$\frac{\partial}{\partial t}\left[\phi_m(p_m^\epsilon)\rho(p_m^\epsilon)\right] - \epsilon^2\nabla\cdot\left[\alpha(p_m^\epsilon)k_m\nabla\Phi_m^\epsilon\right] = 0, \quad x\in\Omega_m^\epsilon, \tag{3.4a}$$

$$p_m^\epsilon = \psi(\Phi_m^\epsilon + x_{c,3}^\epsilon + \epsilon^{-1}(x_3 - x_{c,3}^\epsilon)), \quad x\in\Omega_m^\epsilon, \tag{3.4b}$$

$$\Phi_m^\epsilon = \Phi_f^\epsilon - \Phi_{\rm ref}^\epsilon, \quad x\in\partial\Omega_m^\epsilon. \tag{3.4c}$$

We also need to define the pseudo-potential reference value $\Phi_{\rm ref}^\epsilon$ on each matrix block $\mathcal{Q}_m^\epsilon(x)$. This turns out not to be critical, since any reasonable definition gives rise to the macroscopic model presented below. Consequently, for a given Φ_f^ϵ, let us simply define $\Phi_{\rm ref}^\epsilon$ on $\mathcal{Q}_m^\epsilon(x)$ such that

$$\bar\Phi_f^\epsilon = \frac{1}{|\partial\mathcal{Q}_m^\epsilon|}\int_{\partial\mathcal{Q}_m^\epsilon(x)}\Phi_f^\epsilon(X,t)\,d\sigma(X), \tag{3.5a}$$

$$\begin{aligned}\int_{\mathcal{Q}_m^\epsilon(x)}&\left(\phi_m\rho\right)\left(\psi(\bar\Phi_f^\epsilon - \Phi_{\rm ref}^\epsilon + x_{c,3}^\epsilon(x) + \epsilon^{-1}(X_3 - x_{c,3}^\epsilon(x)))\right)dX\\ &= \int_{\mathcal{Q}_m^\epsilon(x)}\left(\phi_m\rho\right)\left(\psi(\bar\Phi_f^\epsilon + X_3)\right)dX.\end{aligned} \tag{3.5b}$$

So $\Phi_{\rm ref}^\epsilon$ is a function of $\bar\Phi_f^\epsilon(t)$ and $x_{c,3}^\epsilon(x)$.

Our four properties are reflected in this family of governing equations. When $\epsilon = 1$, we have imposed Darcy flow over the medium in the standard way, since then $\Phi_{\rm ref}^\epsilon = 0$, (3.3c) enforces continuity of the mass flux across $\partial\Omega_m^\epsilon$, (3.4c) enforces continuity of the pressures (or pseudo-potentials) across $\partial\Omega_m^\epsilon$, and (3.3b), (3.4b) define the pressures from the pseudo-potentials and the depth. Thus (P1) holds.

The matrix equations have been scaled so that (P2) holds. To see this on $\mathcal{Q}_m^\epsilon(x)$, let $x = x_{c,3}^\epsilon(x) + \epsilon y$ and $\tilde\Phi_m^\epsilon(y,t) = \Phi_m^\epsilon(x,t)$ for $y\in\mathcal{Q}_m$. Then $\nabla\tilde\Phi_m^\epsilon = \epsilon\nabla\Phi_m^\epsilon$, and (3.4a)–(3.4b) is simply unscaled Darcy flow:

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\phi_m(\tilde p_m^\epsilon)\rho(\tilde p_m^\epsilon)\right] - \nabla\cdot\left[\alpha(\tilde p_m^\epsilon)k_m\nabla\tilde\Phi_m^\epsilon\right] = 0, \quad y\in\mathcal{Q}_m,\\ &\tilde p_m^\epsilon = \psi(\tilde\Phi_m^\epsilon + x_{c,3}^\epsilon + y_3), \quad y\in\mathcal{Q}_m.\end{aligned}$$

We have actually scaled k_m by ϵ^2 in (3.3c) and (3.4a). If there is no such scaling, fluid flows more rapidly between the fractures and the matrix system as a whole

when $\epsilon < 1$, because there are more, smaller blocks. In that case, the microscopic models for $\epsilon < 1$ cannot be expected to approximate the $\epsilon = 1$ case. However, if the matrix permeability decreases at some appropriate rate, the matrix-to-fracture flow rate can be maintained. A dimensional argument shows that ϵ^2 is the correct factor [7,6,11]. We also require that the matrix pseudo-potential be defined in (3.4b) by recognizing that the depth of a point in the matrix must be defined on both the intermediate and large scales; depth in the fracture pseudo-potential (3.3b) is defined on the large scale only. Note that then (3.4c) represents a scaled continuity of the pseudo-potentials.

If the fracture system is in gravitational equilibrium, i.e., Φ_f^ϵ is constant, the same is true of the matrix boundary condition, so (P3) holds. Pressure varies around the block, since in this case (3.4b) gives the matrix boundary condition as

$$p_m^\epsilon = \psi\big(\Phi_f^\epsilon - \Phi_{\text{ref}}^\epsilon + x_{c,3}^\epsilon + \epsilon^{-1}(x_3 - x_{c,3}^\epsilon)\big),$$

which is the gravitational equilibrium pressure distribution for some constant $\Phi_{\text{ref}}^\epsilon$ that may vary from matrix block to matrix block. These constants are set by (3.5) which reflects (P4).

We should comment on the solvability of (3.5). In the (uninteresting) case of an incompressible rock and fluid, the pseudo-potential reference is undefined and immaterial, so set $\Phi_{\text{ref}}^\epsilon = 0$. Otherwise, consider the function $(\phi_m\rho)(\psi(\,\cdot\,))$. This function is strictly monotonically increasing and continuous; therefore, for $\Phi_1 = \max_{X\in\mathcal{Q}_m^\epsilon(x)}(\epsilon^{-1}-1)(X_3 - x_{c,3}^\epsilon(x))$ and $\Phi_2 = \min_{X\in\mathcal{Q}_m^\epsilon(x)}(\epsilon^{-1}-1)(X_3 - x_{c,3}^\epsilon(x))$,

$$\begin{aligned}
&\int_{\mathcal{Q}_m^\epsilon(x)} \big(\phi_m\rho\big)\big(\psi(\bar\Phi_f^\epsilon - \Phi_1 + x_{c,3}^\epsilon(x) + \epsilon^{-1}(X_3 - x_{c,3}^\epsilon(x)))\big)\,dX \\
&\quad\le \int_{\mathcal{Q}_m^\epsilon(x)} \big(\phi_m\rho\big)\big(\psi(\bar\Phi_f^\epsilon + X_3)\big)\,dX \\
&\quad\le \int_{\mathcal{Q}_m^\epsilon(x)} \big(\phi_m\rho\big)\big(\psi(\bar\Phi_f^\epsilon - \Phi_2 + x_{c,3}^\epsilon(x) + \epsilon^{-1}(X_3 - x_{c,3}^\epsilon(x)))\big)\,dX.
\end{aligned}$$

Consequently, there is a unique $\Phi_{\text{ref}}^\epsilon$ between Φ_1 and Φ_2 solving (3.5b).

The macroscopic model that results from homogenization ($\epsilon \to 0$) follows. An early version of it appeared in [4]. For the fracture system, we obtain for the fracture pseudo-potential $\Phi_f^0(x,t)$ that

$$\begin{aligned}
\text{(3.6a)}\qquad &\frac{\partial}{\partial t}\big[\phi_f(p_f^0)\rho(p_f^0)\big] + \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_m}\frac{\partial}{\partial t}\big[\phi_m(p_m^0)\rho(p_m^0)\big]\,dy \\
&\qquad - \nabla\cdot\big[\alpha(p_f^0)k_f\nabla\Phi_f^0\big] = 0, \quad x\in\Omega, \\
\text{(3.6b)}\qquad &p_f^0 = \psi(\Phi_f^0 + x_3), \quad x\in\Omega,
\end{aligned}$$

where (2.1) defines ϕ_f and k_f is defined below in (4.12) and (4.17). For the matrix, for each $x\in\Omega$ there is a block congruent to $\mathcal{Q}_m$ on which $\Phi_m^0(x,y,t)$ is determined

from

$$\frac{\partial}{\partial t}\left[\phi_m(p_m^0)\rho(p_m^0)\right] - \nabla_y \cdot \left[\alpha(p_m^0)k_m \nabla_y \Phi_m^0\right] = 0, \quad y \in \mathcal{Q}_m, \tag{3.7a}$$

$$p_m^0 = \psi(\Phi_m^0 + x_3 + y_3), \quad y \in \mathcal{Q}_m, \tag{3.7b}$$

$$\Phi_m^0 = \Phi_f^0 - \Phi_{\text{ref}}^0, \quad y \in \partial\mathcal{Q}_m. \tag{3.7c}$$

In this equation, x is merely a parameter identifying the overall location within the medium, the equations are posed in terms of $y \in \mathcal{Q}_m$, and ∇_y is the gradient with respect to the y-variable. Finally, for a given Φ_f^0 and depth x_3, we define $\Phi_{\text{ref}}^0(\Phi_f^0, x_3)$ such that

$$\frac{1}{|\mathcal{Q}_m|}\int_{\mathcal{Q}_m} \left(\phi_m\rho\right)\left(\psi(\Phi_f^0 - \Phi_{\text{ref}}^0 + x_3 + y_3)\right) dy = \phi_m(p_f^0)\rho(p_f^0). \tag{3.8}$$

Again, the monotonicity of $\phi_m\rho$ insures a unique solution to (3.8) unless the rock and fluid are incompressible. In that case, set $\Phi_{\text{ref}}^0 = 0$. Note that for slightly compressible rock and fluid, $\Phi_{\text{ref}}^0 \approx 0$.

This model says that the fracture system, being highly permeable, rapidly comes into equilibrium on the fracture spacing scale locally. This equilibrium is defined in terms of the pseudo-potential, and it is reflected in the matrix equations through the boundary condition (3.7c). Note also that mass is conserved between the matrix and fracture systems, since fluid flow out of the matrix shows up in the fractures through the integral in (3.6a).

We end this section by reviewing three previous models for a single component in a single phase [7, 6, 11]. First, if no ϵ-scaling is made to (3.3)–(3.4), and (3.5) is omitted, then homogenization produces not a dual-porosity but a single porosity macroscopic model (see, e.g., the two-phase results of [1, 10]) of the form:

$$\frac{\partial}{\partial t}\left[\bar{\phi}(p^0)\rho(p^0)\right] - \nabla \cdot \left[\alpha(p^0)\bar{k}\nabla\Phi^0\right] = 0, \quad x \in \Omega, \tag{3.9a}$$

$$p^0 = \psi(\Phi^0 + x_3), \quad x \in \Omega, \tag{3.9b}$$

where $\bar{\phi} = (\phi_f^*|\mathcal{Q}_f| + \phi_m|\mathcal{Q}_m|)/|\mathcal{Q}|$ is the average porosity and $\bar{k}$ is some average permeability. It is well known that this model is inadequate to capture important features of a fractured porous medium (see, e.g., [20] and Part II of this work [5]).

For the second model, start with a microscopic model that omits (3.5) and defines depth on the large scale only, so that (3.4b)–(3.4c) becomes

$$p_m^\epsilon = \psi(\Phi_m^\epsilon + x_3), \quad x \in \Omega_m^\epsilon, \tag{3.10a}$$

$$p_m^\epsilon = p_f^\epsilon, \quad x \in \partial\Omega_m^\epsilon, \tag{3.10b}$$

where the latter represents continuity of the pressures. Then we obtain the macroscopic dual-porosity model (3.6)–(3.7), except that y_3 is absent from (3.7b) and

$\Phi^0_{\rm ref} = 0$. Since $\nabla_y x_3 = 0$, $\alpha(p^0_m)k_m\nabla_y\Phi^0_m = \mu^{-1}(\rho(p^0_m))^2 k_m \nabla_y p^0_m$, and gravitational effects are missing from the matrix equations. The important boundary condition (3.7c) reads simply as

$$p^0_m = p^0_f, \quad y \in \partial\mathcal{Q}_m. \tag{3.11}$$

The third previous model arises by omitting (3.5) and replacing (3.4c) by (3.10b). Then the macroscopic model is (3.6)–(3.7) except that (3.7c) is replaced by (3.11). Gravity appears in the matrix equation. Unfortunately, this model has a peculiar steady-state, gravitational "equilibrium." If the two time derivative terms are zero and there is no external flux on $\partial\Omega$ into the fractures, then the matrix and fracture fluxes should be zero, i.e., the pseudo-potentials should be constant. If Φ^0_f is constant, Φ^0_m cannot be constant because of (3.11). As a consequence, the equations predict that some perpetual circulation of fluid occurs in the matrix.

The second and third previous models have been presented in [2, 6, 11], and they arose by considering the equations in terms on the pressures rather that the pseudo-potentials. The full model (3.6)–(3.8) has both gravity in the matrix equations and a matrix boundary condition which is consistent with this fact.

4. Formal Homogenization of Single Phase, Single Component Flow.

We now consider the formal homogenization of the system (3.3)–(3.5). We use many of the techniques of [21, 15, 7, 11]. We begin by recalling the asymptotic scaling relations

$$x = x^\epsilon_c(x) + \epsilon y \quad \text{and} \quad \nabla \sim \epsilon^{-1}\nabla_y + \nabla_x. \tag{4.1}$$

For some functions $\Phi^\ell_f(x,y,t)$ and $\Phi^\ell_m(x,y,t)$, $\ell = 0, 1, 2, ...$, we assume the formal asymptotic expansions

$$\Phi^\epsilon_f(x,t) \sim \sum_{\ell=0}^{\infty} \epsilon^\ell \Phi^\ell_f(x,y,t), \quad x \in \Omega,\ y \in \mathcal{Q}_f, \tag{4.2a}$$

$$\Phi^\epsilon_m(x,t) \sim \sum_{\ell=0}^{\infty} \epsilon^\ell \Phi^\ell_m(x,y,t), \quad x \in \Omega,\ y \in \mathcal{Q}_m, \tag{4.2b}$$

where the Φ^ℓ_f are periodic in y with period $\mathcal{Q}_f$, reflecting the periodicity and connectivity of the medium.

We note that if some smooth function F depends on $\pi^\epsilon(x) \sim \sum_{\ell=0}^\infty \epsilon^\ell \pi^\ell(x,y)$, then by expansion about π^0, Taylor's Theorem shows that

$$F(\pi^\epsilon) \sim F\left(\sum_{\ell=0}^{\infty} \epsilon^\ell \pi^\ell\right) = F(\pi^0) + \sum_{\ell=1}^{\infty} \epsilon^\ell F^\ell, \tag{4.3}$$

for some F^ℓ that depend on the π^ℓ's (e.g., $F^1 = F'(\pi^0)\pi^1$). Also, by the change of variables $X = x^\epsilon_c(x) + \epsilon y$,

$$\begin{aligned}\int_{\mathcal{Q}^\epsilon_m(x)} F(\pi^\epsilon(X))\,dX &= \epsilon^3 \int_{\mathcal{Q}_m} F(\pi^\epsilon(x^\epsilon_c(x) + \epsilon y))\,dy \\ &\sim \epsilon^3 \int_{\mathcal{Q}_m} \left[F(\pi^0(x,y)) + \sum_{\ell=1}^{\infty} \epsilon^\ell F^\ell(x,y)\right] dy.\end{aligned} \tag{4.4}$$

Substituting the formal expansions (4.1)–(4.2) into (3.3)–(3.5) and isolating coefficients of powers of ϵ yield an infinite number of equations for the Φ_f^ℓ and Φ_m^ℓ. We analyze the first few equations below.

An application of (4.3) to (3.3b) yields

$$p_f^\epsilon(x,t) \sim \psi(\Phi_f^0(x,y,t) + x_3) + \sum_{\ell=1}^{\infty} \epsilon^\ell \psi^\ell(x,y,t), \tag{4.5}$$

so that in fact $p_f^\epsilon(x,t) \sim \sum_{\ell=0}^{\infty} \epsilon^\ell p_f^\ell(x,y,t)$ can be expanded as in (4.2a), with the $p_f^\ell(x,y,t)$ periodic in y. The ϵ^0 terms define

$$p_f^0(x,y,t) = \psi(\Phi_f^0(x,y,t) + x_3), \quad x \in \Omega,\ y \in \mathcal{Q}_f. \tag{4.6}$$

We will soon see that p_f^0 is independent of y, so that (3.6b) holds. In an entirely similar way, we analyze (3.4b). Since $x_{c,3}^\epsilon + \epsilon^{-1}(x_3 - x_{c,3}^\epsilon) = x_3 + y_3 - \epsilon y_3$, we conclude that we can expand $p_m^\epsilon(x,t) \sim \sum_{\ell=0}^{\infty} \epsilon^\ell p_m^\ell(x,y,t)$, and that (3.7b) holds. It is now trivial to obtain (3.7a) from the ϵ^0-terms of (3.4a). Also, (3.4c) defines the expansion $\Phi_{\text{ref}}^\epsilon \sim \sum_{\ell=0}^{\infty} \epsilon^\ell (\Phi_f^\ell - \Phi_m^\ell)$, from which (3.7c) follows.

The ϵ^{-2} terms of (3.3a), together with the ϵ^{-1} terms of (3.3c) are, for $x \in \Omega$,

$$-\nabla_y \cdot \left[\alpha(p_f^0) k_f^* \nabla_y \Phi_f^0\right] = 0, \quad y \in \mathcal{Q}_f, \tag{4.7a}$$

$$\alpha(p_f^0) k_f^* \nabla_y \Phi_f^0 \cdot \nu = 0, \quad y \in \partial\mathcal{Q}_m. \tag{4.7b}$$

Since Φ_f^0 and p_f^0 are periodic across $\partial\mathcal{Q}$, this equation implies that Φ_f^0 is a constant with respect to y. To see this, simply multiply (4.7a) by Φ_f^0, integrate over $y \in \mathcal{Q}_f$, apply one of the Green's identities (i.e., integrate by parts), and note that the boundary integral is zero by periodicity:

$$\begin{aligned} 0 &= -\int_{\mathcal{Q}_f} \nabla_y \cdot \left[\alpha(p_f^0) k_f^* \nabla_y \Phi_f^0\right] \Phi_f^0 \, dy \\ &= \int_{\mathcal{Q}_f} \alpha(p_f^0) k_f^* \nabla_y \Phi_f^0 \cdot \nabla_y \Phi_f^0 \, dy - \int_{\partial\mathcal{Q}_f} \alpha(p_f^0) k_f^* \nabla_y \Phi_f^0 \cdot \nu_{\mathcal{Q}_f} \, \Phi_f^0 \, d\sigma(y) \\ &= \int_{\mathcal{Q}_f} \alpha(p_f^0) k_f^* \nabla_y \Phi_f^0 \cdot \nabla_y \Phi_f^0 \, dy \geq 0. \end{aligned} \tag{4.8}$$

Hence $\nabla_y \Phi_f^0 = 0$ and we write that

$$\Phi_f^0 = \Phi_f^0(x,t) \quad \text{and} \quad p_f^0 = p_f^0(x,t) \quad \text{only}, \tag{4.9}$$

where the latter conclusion uses (4.6). Thus the macroscopic fracture flow is defined on the large scale only.

A result analogous to (4.4) for boundary integrals applied to (3.5a) implies an expansion for $\bar{\Phi}_f^\epsilon$; moreover, $\bar{\Phi}_f^0 = |\partial\mathcal{Q}_m|^{-1}\int_{\partial\mathcal{Q}_m}\Phi_f^0\,d\sigma(y) = \Phi_f^0$. Now (4.4) applied to (3.5b) gives that

$$\epsilon^3\int_{\mathcal{Q}_m}(\phi_m\rho)\big(\psi(\bar{\Phi}_f^\epsilon - \Phi_{\rm ref}^\epsilon + x_{c,3}^\epsilon(x)+y_3)\big)\,dy = \epsilon^3\int_{\mathcal{Q}_m}(\phi_m\rho)\big(\psi(\bar{\Phi}_f^\epsilon + x_{c,3}^\epsilon(x)+\epsilon y_3)\big)\,dy. \tag{4.10}$$

Since $x_{c,3}^\epsilon(x) = x_3 - \epsilon y_3$, the ϵ^0-terms can be extracted with (4.3), and then (3.6b) gives (3.8). It remains only to show that (3.6a) holds.

Using (4.9) to delete y-derivatives of Φ_f^0 and p_f^0, the ϵ^{-1} terms of (3.3a) and the ϵ^0 terms of (3.3c) are, for $x\in\Omega$,

$$-\nabla_y\cdot\big[\alpha(p_f^0)k_f^*\nabla_y\Phi_f^1\big] = 0,\quad y\in\mathcal{Q}_f, \tag{4.11a}$$

$$\alpha(p_f^0)k_f^*\big(\nabla_y\Phi_f^1 + \nabla_x\Phi_f^0\big)\cdot\nu = 0,\quad y\in\partial\mathcal{Q}_m. \tag{4.11b}$$

This is a linear elliptic problem for Φ_f^1 in terms of Φ_f^0. To solve it, find $\omega_j(y)$, $j=1,2,3$, such that

$$-\nabla_y\cdot[k_f^*\nabla_y\omega_j] = 0,\quad y\in\mathcal{Q}_f, \tag{4.12a}$$

$$k_f^*\nabla_y\omega_j\cdot\nu = -k_f^*e_j\cdot\nu,\quad y\in\partial\mathcal{Q}_m, \tag{4.12b}$$

where the ω_j are periodic across $\partial\mathcal{Q}$, and then

$$\Phi_f^1 = \sum_{j=1}^3\frac{\partial\Phi_f^0}{\partial x_j}\omega_j + \pi, \tag{4.13}$$

for some $\pi(x)$ independent of y.

Finally, the ϵ^0 and ϵ^1 terms of (3.3a) and (3.3c) are

$$\begin{aligned}&\frac{\partial}{\partial t}\big[\phi_f^*(p_f^0)\rho(p_f^0)\big]\\ &\quad-\nabla_y\cdot\big[\alpha(p_f^0)k_f^*\big(\nabla_y\Phi_f^2+\nabla_x\Phi_f^1\big)+\alpha^1k_f^*\big(\nabla_y\Phi_f^1+\nabla_x\Phi_f^0\big)\big]\\ &\quad-\nabla_x\cdot\big[\alpha(p_f^0)k_f^*\big(\nabla_y\Phi_f^1+\nabla_x\Phi_f^0\big)\big] = 0,\quad x\in\Omega,\ y\in\mathcal{Q}_f,\end{aligned} \tag{4.14a}$$

$$\begin{aligned}&\big[\alpha(p_f^0)k_f^*\big(\nabla_y\Phi_f^2+\nabla_x\Phi_f^1\big)+\alpha^1k_f^*\big(\nabla_y\Phi_f^1+\nabla_x\Phi_f^0\big)\big]\cdot\nu\\ &\quad=\alpha(p_m^0)k_m\nabla_y\Phi_m^0\cdot\nu,\quad x\in\Omega,\ y\in\partial\mathcal{Q}_m,\end{aligned} \tag{4.14b}$$

where (4.3) defines $\alpha^1 = \alpha'(p_f^0)p_f^1$. We remove the y dependence in (4.14a) by macroscopically averaging over the cell $\mathcal{Q}_f$, i.e., by integrating over $\mathcal{Q}_f$ and dividing

by $|\mathcal{Q}|$. By the Divergence Theorem, (4.14b), the Divergence Theorem again, and (3.7a), the second term becomes

$$
\begin{aligned}
(4.15)\qquad & -\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f}\nabla_y\cdot\left[\alpha(p_f^0)k_f^*(\nabla_y\Phi_f^2+\nabla_x\Phi_f^1)+\alpha^1 k_f^*(\nabla_y\Phi_f^1+\nabla_x\Phi_f^0)\right]dy \\
&\quad=-\frac{1}{|\mathcal{Q}|}\int_{\partial\mathcal{Q}_f}\left[\alpha(p_f^0)k_f^*(\nabla_y\Phi_f^2+\nabla_x\Phi_f^1)\right. \\
&\qquad\qquad\left.+\alpha^1 k_f^*(\nabla_y\Phi_f^1+\nabla_x\Phi_f^0)\right]\cdot\nu_{\mathcal{Q}_f}\,d\sigma(y) \\
&\quad=\frac{1}{|\mathcal{Q}|}\int_{\partial\mathcal{Q}_m}\alpha(p_m^0)k_m\nabla_y\Phi_m^0\cdot\nu\,d\sigma(y) \\
&\quad=\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_m}\nabla_y\cdot\left[\alpha(p_m^0)k_m\nabla_y\Phi_m^0\right]dy \\
&\quad=\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_m}\frac{\partial}{\partial t}\left[\phi_m(p_m^0)\rho(p_m^0)\right]dy.
\end{aligned}
$$

The second equality above used the periodicity of the Φ_f^ℓ and p_f^ℓ to handle $\partial\mathcal{Q}\subset\partial\mathcal{Q}_f$ and the fact that the normal vectors reversed direction as we went from $\mathcal{Q}_f$ to $\mathcal{Q}_m$. This term is the macroscopic, distributed matrix source (i.e., the "transfer function").

The third term in (4.14a) can be rewritten with (4.13) as

$$
\begin{aligned}
(4.16)\qquad & -\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f}\nabla_x\cdot\left[\alpha(p_f^0)k_f^*(\nabla_y\Phi_f^1+\nabla_x\Phi_f^0)\right]dy \\
&\quad=-\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f}\nabla_x\cdot\left[\alpha(p_f^0)k_f^*\left(\sum_{j=1}^3\frac{\partial\Phi_f^0}{\partial x_j}\nabla_y\omega_j+\nabla_x\Phi_f^0\right)\right]dy \\
&\quad=-\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f}\sum_{i=1}^3\frac{\partial}{\partial x_i}\left[\alpha(p_f^0)\sum_{k=1}^3 k_{f,ik}^*\left(\sum_{j=1}^3\frac{\partial\Phi_f^0}{\partial x_j}\left(\frac{\partial\omega_j}{\partial y_k}+\delta_{kj}\right)\right)\right]dy \\
&\quad=-\nabla_x\cdot\left[\alpha(p_f^0)k_f\nabla_x\Phi_f^0\right],
\end{aligned}
$$

where δ_{kj} is the Kronecker delta and we define the tensor k_f by

$$
(4.17)\qquad k_{f,ij}=\sum_{k=1}^3\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f}k_{f,ik}^*\left(\frac{\partial\omega_j}{\partial y_k}+\delta_{kj}\right)dy.
$$

It is known that this definition results in k_f being symmetric and positive definite (see, e.g., [6]).

Combining (4.14a) with (4.15) and (4.16), we obtain (3.6a), and the derivation of the macroscopic model is complete. The argument of this section can be made completely mathematically rigorous if the problem is linearized, see [6].

5. Two-Phase, Completely Immiscible Flow.

In this section we consider the flow problem for two fluids when they form two distinct phases. Results completely analogous to the previous two sections are obtained.

Subscript 1 refers to the wetting phase, and 2 to the nonwetting phase. Denote by s_i and $k_{ri}(s_1)$, $i = 1, 2$, the phase saturation and relative permeabilities, respectively. Let $p_c(s_1)$ denote the capillary pressure. Analogous to (3.1), we can define the pseudo-potential for each phase as

$$\Phi_i = \int_{p_{0,i}}^{p_i} \frac{d\pi}{\rho_i(\pi)g} - x_3, \tag{5.1}$$

for some reference pressure $p_{0,i}$. We define ψ_i so that $p_i = \psi_i(\Phi_i + x_3)$; moreover, $\psi_i' = \rho_i(\psi_i)g$.

The microscopic model of Darcy flow over the entire medium on the fine scale follows, wherein $\alpha_i(p) = (\rho_i(p))^2 g/\mu_i$. For each fracture phase $i = 1, 2$,

$$\frac{\partial}{\partial t}\left[\phi_f^*(p_{1,f}^\epsilon)\rho_i(p_{i,f}^\epsilon)s_{i,f}^\epsilon\right] - \nabla\cdot\left[\alpha_i(p_{i,f}^\epsilon)k_f^* k_{ri,f}(s_{1,f}^\epsilon)\nabla\Phi_{i,f}^\epsilon\right] = 0, \quad x \in \Omega_f^\epsilon, \tag{5.2a}$$

$$p_{i,f}^\epsilon = \psi_i(\Phi_{i,f}^\epsilon + x_3), \quad x \in \Omega_f^\epsilon, \tag{5.2b}$$

$$s_{1,f}^\epsilon + s_{2,f}^\epsilon = 1, \quad x \in \Omega_f^\epsilon, \tag{5.2c}$$

$$p_{c,f}(s_{1,f}^\epsilon) = p_{2,f}^\epsilon - p_{1,f}^\epsilon, \quad x \in \Omega_f^\epsilon, \tag{5.2d}$$

$$\alpha(p_{i,f}^\epsilon)k_f^* k_{ri,f}(s_{1,f}^\epsilon)\nabla\Phi_{i,f}^\epsilon\cdot\nu = \epsilon^2\alpha(p_{i,m}^\epsilon)k_m k_{ri,m}(s_{1,m}^\epsilon)\nabla\Phi_{i,m}^\epsilon\cdot\nu, \quad x \in \partial\Omega_m^\epsilon. \tag{5.2e}$$

For the matrix, for $i = 1, 2$,

$$\frac{\partial}{\partial t}\left[\phi_m(p_{1,m}^\epsilon)\rho_i(p_{i,m}^\epsilon)s_{i,m}^\epsilon\right] - \epsilon^2\nabla\cdot\left[\alpha_i(p_{i,m}^\epsilon)k_m k_{ri,m}(s_{1,m}^\epsilon)\nabla\Phi_{i,m}^\epsilon\right] = 0, \quad x \in \Omega_m^\epsilon, \tag{5.3a}$$

$$p_{i,m}^\epsilon = \psi_i(\Phi_{i,m}^\epsilon + x_{c,3}^\epsilon(x) + \epsilon^{-1}(x_3 - x_{c,3}^\epsilon(x))), \quad x \in \Omega_m^\epsilon, \tag{5.3b}$$

$$s_{1,m}^\epsilon + s_{2,m}^\epsilon = 1, \quad x \in \Omega_m^\epsilon, \tag{5.3c}$$

$$p_{c,m}(s_{1,m}^\epsilon) = p_{2,m}^\epsilon - p_{1,m}^\epsilon, \quad x \in \Omega_m^\epsilon, \tag{5.3d}$$

$$\Phi_{i,m}^\epsilon = \Phi_{i,f}^\epsilon - \Phi_{\mathrm{ref},i}^\epsilon, \quad x \in \partial\Omega_m^\epsilon. \tag{5.3e}$$

Define the $\Phi^\epsilon_{\mathrm{ref},i}$, $i = 1, 2$, on $\mathcal{Q}^\epsilon_m(x)$ for given $\Phi^\epsilon_{i,f}$ by the following:

$$\bar{\Phi}^\epsilon_{i,f} = \frac{1}{|\partial\mathcal{Q}^\epsilon_m|}\int_{\partial\mathcal{Q}^\epsilon_m(x)} \Phi^\epsilon_{i,f}(X,t)\,d\sigma(X), \tag{5.4a}$$

$$\tilde{p}^\epsilon_i(X) = \psi_i(\bar{\Phi}^\epsilon_{i,f} - \Phi^\epsilon_{\mathrm{ref},i} + x^\epsilon_{c,3}(x) + \epsilon^{-1}(X_3 - x^\epsilon_{c,3}(x))), \tag{5.4b}$$

$$\bar{p}^\epsilon_i(X) = \psi_i(\bar{\Phi}^\epsilon_{i,f} + X_3), \tag{5.4c}$$

$$\tilde{s}^\epsilon_1(X) = p^{-1}_{c,m}(\tilde{p}^\epsilon_2 - \tilde{p}^\epsilon_1) \quad \text{and} \quad \tilde{s}^\epsilon_2(X) = 1 - \tilde{s}^\epsilon_1, \tag{5.4d}$$

$$\bar{s}^\epsilon_1(X) = p^{-1}_{c,m}(\bar{p}^\epsilon_2 - \bar{p}^\epsilon_1) \quad \text{and} \quad \bar{s}^\epsilon_2(X) = 1 - \bar{s}^\epsilon_1, \tag{5.4e}$$

$$\int_{\mathcal{Q}^\epsilon_m(x)} \phi_m(\tilde{p}^\epsilon_1)\rho_i(\tilde{p}^\epsilon_i)\tilde{s}^\epsilon_i\,dX = \int_{\mathcal{Q}^\epsilon_m(x)} \phi_m(\bar{p}^\epsilon_1)\rho_i(\bar{p}^\epsilon_i)\bar{s}^\epsilon_i\,dX. \tag{5.4f}$$

It is not obvious that (5.4) has a unique solution. We show this to be true in the Appendix.

As in Section 3, our properties (P1)–(P4) are reflected in this family (5.2)–(5.4) of microscopic governing equations.

The macroscopic model that results from homogenization follows. For each fracture phase $i = 1, 2$, for $\Phi^0_{i,f}(x,t)$ we have that

$$\frac{\partial}{\partial t}\left[\phi_f(p^0_{1,f})\rho_i(p^0_{i,f})s^0_{i,f}\right] + \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_m}\frac{\partial}{\partial t}\left[\phi_m(p^0_{1,m})\rho_i(p^0_{i,m})s^0_{i,m}\right]dy$$
$$- \nabla\cdot\left[\alpha_i(p^0_{i,f})k_f k_{ri,f}(s^0_{1,f})\nabla\Phi^0_{i,f}\right] = 0, \quad x \in \Omega, \tag{5.5a}$$

$$p^0_{i,f} = \psi_i(\Phi^0_{i,f} + x_3), \quad x \in \Omega, \tag{5.5b}$$

$$s^0_{1,f} + s^0_{2,f} = 1, \quad x \in \Omega, \tag{5.5c}$$

$$p_{c,f}(s^0_{1,f}) = p^0_{2,f} - p^0_{1,f}, \quad x \in \Omega, \tag{5.5d}$$

where ϕ_f and k_f are defined in (2.1), (4.12), and (4.17). For the matrix, for each $x \in \Omega$ and $i = 1, 2$, we have for $\Phi^0_{i,m}(x,y,t)$ that

$$\frac{\partial}{\partial t}\left[\phi_m(p^0_{1,m})\rho_i(p^0_{i,m})s^0_{i,m}\right]$$
$$- \nabla_y\cdot\left[\alpha_i(p^0_{i,m})k_m k_{ri,m}(s^0_{1,m})\nabla_y\Phi^0_{i,m}\right] = 0, \quad y \in \mathcal{Q}_m, \tag{5.6a}$$

$$p^0_{i,m} = \psi_i(\Phi^0_{i,m} + x_3 + y_3), \quad y \in \mathcal{Q}_m, \tag{5.6b}$$

$$s^0_{1,m} + s^0_{2,m} = 1, \quad y \in \mathcal{Q}_m, \tag{5.6c}$$

$$p_{c,m}(s^0_{1,m}) = p^0_{2,m} - p^0_{1,m}, \quad y \in \mathcal{Q}_m, \tag{5.6d}$$

$$\Phi^0_{i,m} = \Phi^0_{i,f} - \Phi^0_{\mathrm{ref},i}, \quad y \in \partial\mathcal{Q}_m. \tag{5.6e}$$

Define $\Phi^0_{\mathrm{ref},i}(\Phi^0_{i,f}, x_3)$, $i = 1, 2$, for given $\Phi^0_{i,f}$ and x_3 by

$$\tilde{p}^0_i(y) = \psi_i(\Phi^0_{i,f} - \Phi^0_{\mathrm{ref},i} + x_3 + y_3), \tag{5.7a}$$

$$\tilde{s}^0_1(y) = p^{-1}_{c,m}(\tilde{p}^0_2 - \tilde{p}^0_1) \quad \text{and} \quad \tilde{s}^0_2(y) = 1 - \tilde{s}^0_1, \tag{5.7b}$$

$$\frac{1}{|\mathcal{Q}_m|}\int_{\mathcal{Q}_m}\phi_m(\tilde{p}^0_1)\rho_i(\tilde{p}^0_i)\tilde{s}^0_i\,dy = \phi_m(p^0_{1,f})\rho_i(p^0_{i,f})s^0_{i,f}. \tag{5.7c}$$

The question of the solvability of (5.7) is considered in the Appendix. Note that $\Phi^0_{\mathrm{ref},i}$ must vary considerably with time on each matrix block even if the rock and fluids are incompressible.

The formal homogenization of the system (5.2)–(5.4) proceeds as in the previous section, except that we must be careful to properly account for the degeneracy of the relative permeabilities. We formally expand $\Phi^\epsilon_{i,f}$ and $\Phi^\epsilon_{i,m}$ in powers of ϵ as in (4.2) above, with the fracture quantities being periodic in y. Arguing in analogy to the single-phase, single component case (see (4.5)–(4.6)), we easily conclude from (5.2b)–(5.2d) and (5.3) that $p^\epsilon_{i,f}$, $s^\epsilon_{i,f}$, $p^\epsilon_{i,m}$, $s^\epsilon_{i,m}$, and $\Phi^\epsilon_{\mathrm{ref},i}$ can be expanded as in (4.2), again with the fracture quantities being periodic in y, and that in fact the ϵ^0-terms satisfy (5.5b)–(5.5d) and (5.6).

As in (4.7)–(4.9) above, we want to conclude that

$$\Phi^0_{i,f} = \Phi^0_{i,f}(x,t), \quad p^0_{i,f} = p^0_{i,f}(x,t), \quad \text{and} \quad s^0_{i,f} = s^0_{i,f}(x,t) \quad \text{only.} \tag{5.8}$$

This can be seen by considering the ϵ^{-2} terms of (5.2a) and the ϵ^{-1} terms of (5.2e), together with (5.5b)–(5.5d). We get that for $i = 1, 2$,

$$-\nabla_y \cdot \left[\alpha(p^0_{i,f}) k^*_f k_{ri,f}(s^0_{1,f}) \nabla_y \Phi^0_{i,f}\right] = 0, \quad y \in \mathcal{Q}_f, \tag{5.9a}$$

$$\alpha(p^0_{i,f}) k^*_f k_{ri,f}(s^0_{1,f}) \nabla_y \Phi^0_{i,f} \cdot \nu = 0, \quad y \in \partial\mathcal{Q}_m, \tag{5.9b}$$

from which we conclude that $\left(k^*_f k_{ri,f}(s^0_{1,f})\right)^{1/2} \nabla_y \Phi^0_{i,f} = 0$. But since $k_{ri,f}(s^0_{1,f})$ may be zero, we must be careful to conclude (5.8) [7, 11]. If $k_{r1,f}(s^0_{1,f}) = 0$ for $y \in \mathcal{Q}_{f,1} \subset \mathcal{Q}_f$, then $s^0_{1,f}$ is at its residual saturation there, which is independent of y. Also then, $k_{r2,f}(s^0_{1,f}) \neq 0$, so $\nabla_y \Phi^0_{2,f} = 0$ for $y \in \mathcal{Q}_{f,1}$. Finally, (5.5b) and (5.5d) give that $\nabla_y \Phi^0_{1,f} = 0$ for $y \in \mathcal{Q}_{f,1}$. We argue similarly if $k_{r2,f}(s^0_{1,f}) = 0$ for $y \in \mathcal{Q}_{f,2} \subset \mathcal{Q}_f$, and more simply if both are nonzero for $y \in \mathcal{Q}_f \backslash \{\mathcal{Q}_{f,1} \cup \mathcal{Q}_{f,2}\}$. Assuming continuity of $\Phi^0_{i,f}$, $\nabla_y \Phi^0_{i,f} = 0$ on all of $\mathcal{Q}_f$, so the $\Phi^0_{i,f}$ are indeed independent of y (since $\mathcal{Q}_f$ is connected). Finally, (5.5b)–(5.5d) give us the same conclusion for the $p^0_{i,f}$ and $s^0_{i,f}$.

We can now obtain (5.7) from (5.4) using (4.3)–(4.4).

The ϵ^{-1} terms of (5.2a) and the ϵ^0 terms of (5.2e) imply that for $i = 1, 2$,

$$-\nabla_y \cdot \left[\alpha(p^0_{i,f}) k^*_f k_{ri,f}(s^0_{1,f}) \nabla_y \Phi^1_{i,f}\right] = 0, \quad y \in \mathcal{Q}_f, \tag{5.10a}$$

$$\alpha(p^0_{i,f}) k^*_f k_{ri,f}(s^0_{1,f}) \left(\nabla_y \Phi^1_{i,f} + \nabla_x \Phi^0_{i,f}\right) \cdot \nu = 0, \quad y \in \partial\mathcal{Q}_m. \tag{5.10b}$$

Using (4.12) above, we conclude that either

$$\Phi^1_{i,f} = \sum_{j=1}^{3} \frac{\partial \Phi^0_{i,f}}{\partial x_j} \omega_j + \pi_i, \tag{5.11}$$

for some $\pi_i(x)$ independent of y, or $k_{ri,f}(s^0_{1,f}) = 0$. This ambiguity does not hinder us in the rest of the argument.

Finally we turn to the ϵ^0 terms of (5.2a) and the ϵ^1 terms of (5.2e). As in the analysis (4.14)–(4.17) of Section 4, use of (5.6a), (5.8), (5.11), and (4.17) lead us to (5.5a) for each phase, and the derivation is complete.

6. Two Component, Completely Miscible Flow.

We consider now two components in a single phase. In this section, density and viscosity depend on the concentrations of the two components. Let c denote the concentration of one of them (the other is then $1-c$). We write the mass balance equation for this selected component, as well as the mass balance equation for the total fluid (i.e., the sum of the two component mass balance equations). We will need the molecular diffusion coefficient in the fractures defined on the scale of the fracture thickness, D_f^*, and the diffusion/dispersion tensor in the matrix, $D_m(u)$, which is a function of the Darcy velocity u.

Since $\rho = \rho(c,p)$, there is no pseudo-potential for this system. However, the equilibrium concentration distribution will be constant, because of molecular diffusion. Analogous to (3.1), we can define the gravitational equilibrium pressure distribution $\psi(\xi, x_3)$ for a given fixed concentration ξ as the solution to $\partial\psi(\xi,x_3)/\partial x_3 = \rho(\xi,\psi(\xi,x_3))g$; that is, for some reference pressure p_0,

$$\int_{p_0}^{\psi(\xi,x_3)} \frac{d\pi}{\rho(\xi,\pi)g} = x_3. \tag{6.1}$$

The inverse of ψ, again for a fixed ξ, is denoted by $\psi^{-1}(\xi,\cdot)$. For any pressure p, $\psi^{-1}(\xi,p) = \int_{p_0}^{p}[\rho(\xi,\pi)g]^{-1}\,d\pi$ is the pseudo-potential plus x_3 under the condition that concentration is in equilibrium.

The microscopic model follows. Again it merely represents Darcy flow over the medium on the scale of the fracture thickness. For the fractures,

$$\begin{aligned}
&\frac{\partial}{\partial t}\big[\phi_f^*(p_f^\epsilon)\rho(c_f^\epsilon,p_f^\epsilon)c_f^\epsilon\big] \\
&\quad + \nabla\cdot\big[\rho(c_f^\epsilon,p_f^\epsilon)\big(c_f^\epsilon u_f^{*,\epsilon} - D_f^*\nabla c_f^\epsilon\big)\big] = 0, \quad x\in\Omega_f^\epsilon,
\end{aligned} \tag{6.2a}$$

$$\frac{\partial}{\partial t}\big[\phi_f^*(p_f^\epsilon)\rho(c_f^\epsilon,p_f^\epsilon)\big] + \nabla\cdot\big[\rho(c_f^\epsilon,p_f^\epsilon)u_f^{*,\epsilon}\big] = 0, \quad x\in\Omega_f^\epsilon, \tag{6.2b}$$

$$u_f^{*,\epsilon} = -\mu(c_f^\epsilon)^{-1}k_f^*\big[\nabla p_f^\epsilon - \rho(c_f^\epsilon,p_f^\epsilon)ge_3\big], \quad x\in\Omega_f^\epsilon, \tag{6.2c}$$

$$\rho(c_f^\epsilon,p_f^\epsilon)D_f^*\nabla c_f^\epsilon\cdot\nu = \epsilon^2\rho(c_m^\epsilon,p_m^\epsilon)D_m(u_m^\epsilon)\nabla c_m^\epsilon\cdot\nu, \quad x\in\partial\Omega_m^\epsilon, \tag{6.2d}$$

$$\rho(c_f^\epsilon,p_f^\epsilon)u_f^{*,\epsilon}\cdot\nu = \epsilon\rho(c_m^\epsilon,p_m^\epsilon)u_m^\epsilon\cdot\nu, \quad x\in\partial\Omega_m^\epsilon. \tag{6.2e}$$

For the matrix,

$$\begin{aligned}
&\frac{\partial}{\partial t}\big[\phi_m(p_m^\epsilon)\rho(c_m^\epsilon,p_m^\epsilon)c_m^\epsilon\big] \\
&\quad + \epsilon\nabla\cdot\big[\rho(c_m^\epsilon,p_m^\epsilon)\big(c_m^\epsilon u_m^\epsilon - \epsilon D_m(u_m^\epsilon)\nabla c_m^\epsilon\big)\big] = 0, \quad x\in\Omega_m^\epsilon,
\end{aligned} \tag{6.3a}$$

$$\frac{\partial}{\partial t}\big[\phi_m(p_m^\epsilon)\rho(c_m^\epsilon,p_m^\epsilon)\big] + \epsilon\nabla\cdot\big[\rho(c_m^\epsilon,p_m^\epsilon)u_m^\epsilon\big] = 0, \quad x\in\Omega_m^\epsilon, \tag{6.3b}$$

$$u_m^\epsilon = -\mu(c_m^\epsilon)^{-1}k_m\big[\epsilon\nabla p_m^\epsilon - \rho(c_m^\epsilon,p_m^\epsilon)ge_3\big], \quad x\in\Omega_m^\epsilon, \tag{6.3c}$$

$$c_m^\epsilon = c_f^\epsilon, \quad x\in\partial\Omega_m^\epsilon, \tag{6.3d}$$

$$p_m^\epsilon = \psi\big(c_m^\epsilon,\psi^{-1}(c_f^\epsilon,p_f^\epsilon) - \Phi_{\mathrm{ref}}^\epsilon + (\epsilon^{-1}-1)(x_3 - x_{c,3}^\epsilon)\big), \quad x\in\partial\Omega_m^\epsilon. \tag{6.3e}$$

The auxiliary condition defining $\Phi^\epsilon_{\text{ref}}$ is given by

$$\bar{c}^\epsilon = \frac{1}{|\partial \mathcal{Q}^\epsilon_m|} \int_{\partial \mathcal{Q}^\epsilon_m(x)} c^\epsilon_f(X,t)\, d\sigma(X), \tag{6.4a}$$

$$\bar{\Phi}^\epsilon = \frac{1}{|\partial \mathcal{Q}^\epsilon_m|} \int_{\partial \mathcal{Q}^\epsilon_m(x)} \left[\psi^{-1}(\bar{c}^\epsilon(t), p^\epsilon_f(X,t)) - X_3\right] d\sigma(X), \tag{6.4b}$$

$$\begin{aligned} &\int_{\mathcal{Q}^\epsilon_m(x)} \big(\phi_m \rho\big)\big(\bar{c}^\epsilon, \psi(\bar{c}^\epsilon, \bar{\Phi}^\epsilon - \Phi^\epsilon_{\text{ref}} + x^\epsilon_{c,3} + \epsilon^{-1}(X_3 - x^\epsilon_{c,3}))\big)\, dX \\ &\quad = \int_{\mathcal{Q}^\epsilon_m(x)} \big(\phi_m \rho\big)\big(\bar{c}^\epsilon, \psi(\bar{c}^\epsilon, \bar{\Phi}^\epsilon + X_3)\big)\, dX. \end{aligned} \tag{6.4c}$$

For fixed concentration c, $(\phi_m\rho)(c, \psi(c, \cdot))$ is monotone and increasing, so a solution exists unless ϕ_m and ρ do not change with pressure, in which case set $\Phi^\epsilon_{\text{ref}} = 0$.

We can easily check that Properties (P1)–(P4) hold.

The homogenized macroscopic model follows. For the fracture system, we solve for $p^0_f(x,t)$ and $c^0_f(x,t)$ such that

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\phi_f(p^0_f)\rho(c^0_f, p^0_f)c^0_f\right] + \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_m} \frac{\partial}{\partial t}\left[\phi_m(p^0_m)\rho(c^0_m, p^0_m)c^0_m\right] dy \\ &\quad + \nabla \cdot \left[\rho(c^0_f, p^0_f)(c^0_f u^0_f - D_f \nabla c^0_f)\right] = 0, \quad x \in \Omega, \end{aligned} \tag{6.5a}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\phi_f(p^0_f)\rho(c^0_f, p^0_f)\right] + \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_m} \frac{\partial}{\partial t}\left[\phi_m(p^0_m)\rho(c^0_m, p^0_m)\right] dy \\ &\quad + \nabla \cdot \left[\rho(c^0_f, p^0_f)u^0_f\right] = 0, \quad x \in \Omega, \end{aligned} \tag{6.5b}$$

$$u^0_f = -\mu(c^0_f)^{-1} k_f \left[\nabla p^0_f - \rho(c^0_f, p^0_f) g e_3\right], \tag{6.5c}$$

where (2.1), (4.12), and (4.17) above, and (6.13) and (6.19) below define ϕ_f, k_f, and D_f. For the matrix we find for $x \in \Omega$, $p^0_m(x,y,t)$ and $c^0_m(x,y,t)$ satisfying

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\phi_m(p^0_m)\rho(c^0_m, p^0_m)c^0_m\right] \\ &\quad + \nabla_y \cdot \left[\rho(c^0_m, p^0_m)(c^0_m u^0_m - D_m(u^0_m)\nabla_y c^0_m)\right] = 0, \quad y \in \mathcal{Q}_m, \end{aligned} \tag{6.6a}$$

$$\frac{\partial}{\partial t}\left[\phi_m(p^0_m)\rho(c^0_m, p^0_m)\right] + \nabla_y \cdot \left[\rho(c^0_m, p^0_m)u^0_m\right] = 0, \quad y \in \mathcal{Q}_m, \tag{6.6b}$$

$$u^0_m = -\mu(c^0_m)^{-1} k_m \left[\nabla_y p^0_m - \rho(c^0_m, p^0_m) g e_3\right], \quad y \in \mathcal{Q}_m, \tag{6.6c}$$

$$c^0_m = c^0_f, \quad y \in \partial \mathcal{Q}_m, \tag{6.6d}$$

$$p^0_m = \psi\big(c^0_m, \psi^{-1}(c^0_f, p^0_f) - \Phi^0_{\text{ref}} + y_3\big), \quad y \in \partial \mathcal{Q}_m. \tag{6.6e}$$

The auxiliary condition defining $\Phi^0_{\text{ref}}(c^0_f, p^0_f)$ is

$$\frac{1}{|\mathcal{Q}_m|}\int_{\mathcal{Q}_m} \big(\phi_m\rho\big)\big(c^0_f, \psi(c^0_f, \psi^{-1}(c^0_f, p^0_f) - \Phi^0_{\text{ref}} + y_3)\big) dy = \phi_m(p^0_f)\rho(c^0_f, p^0_f). \tag{6.7}$$

Again, $\Phi^0_{\text{ref}} \approx 0$ for slightly compressible rock and fluids; if these are incompressible, set $\Phi^0_{\text{ref}} = 0$.

For the formal homogenization, we expand p_f^ϵ, c_f^ϵ, p_m^ϵ, and c_m^ϵ in powers of ϵ as in (4.2), the first two periodic in y. By (6.3c), u_m^ϵ can be expanded as usual, and (6.6c) defines u_m^0. However, (6.2c) implies that for some $u_f^{*,\ell}(x,y,t)$ periodic in y, $u_f^{*,\epsilon} \sim \sum_{\ell=-1}^{\infty} \epsilon^\ell u_f^{*,\ell}$; in fact,

$$u_f^{*,-1} = -\mu(c_f^0)^{-1} k_f^* \nabla_y p_f^0, \tag{6.8a}$$

$$u_f^{*,0} = -\mu(c_f^0)^{-1} k_f^* \big(\nabla_y p_f^1 + \nabla_x p_f^0 - \rho(c_f^0, p_f^0) g e_3\big) - \big(\mu^{-1}\big)^1 k_f^* \nabla_y p_f^0. \tag{6.8b}$$

It is straightforward to obtain the rest of (6.6) from (6.3), noting that $\Phi^\epsilon_{\text{ref}}$ can be expanded in powers of ϵ. After substituting for $u_f^{*,\epsilon}$, the ϵ^{-2} terms of (6.2a)–(6.2b) and the ϵ^{-1} terms of (6.2d)–(6.2e) enable us to conclude easily that

$$p_f^0 = p_f^0(x,t) \quad \text{and} \quad c_f^0 = c_f^0(x,t) \quad \text{only.} \tag{6.9}$$

We now consider (6.4). With the argument of (4.4), we expand $\bar{c}^\epsilon$ to see that $\bar{c}^0 = c_f^0$, and then we use the two variable Taylor Theorem to expand

$$\begin{aligned}\bar{\Phi}^\epsilon &= \frac{1}{|\partial \mathcal{Q}_m|} \int_{\partial \mathcal{Q}_m} \big[\psi^{-1}(\bar{c}^\epsilon(t), p_f^\epsilon(x_c^\epsilon(x) + \epsilon y, t)) - x_{c,3}^\epsilon(x) - \epsilon y_3\big]\, d\sigma(y) \\ &\sim \frac{1}{|\partial \mathcal{Q}_m|} \int_{\partial \mathcal{Q}_m} \big[\psi^{-1}(\bar{c}^0, p_f^0) - x_3 + \sum_{\ell=1}^{\infty} \epsilon^\ell (\psi^{-1})^1\big]\, d\sigma(y)\end{aligned}$$

to see that $\bar{\Phi}^0 = \psi^{-1}(c_f^0, p_f^0) - x_3$. As a consequence, (6.7) follows from (6.4c).

The ϵ^{-1} terms of (6.2b) and the ϵ^0 terms of (6.2e) are

$$-\nabla_y \cdot \big[\mu(c_f^0)^{-1} \rho(c_f^0, p_f^0) k_f^* \nabla_y p_f^1\big] = 0, \quad y \in \mathcal{Q}_f, \tag{6.10a}$$

$$\mu(c_f^0)^{-1} \rho(c_f^0, p_f^0) k_f^* \big(\nabla_y p_f^1 + \nabla_x p_f^0 - \rho(c_f^0, p_f^0) g e_3\big) \cdot \nu = 0, \quad y \in \partial \mathcal{Q}_m, \tag{6.10b}$$

and, recalling (4.12), they easily imply that

$$p_f^1 = \sum_{j=1}^{3} \frac{\partial p_f^0}{\partial x_j} \omega_j - \rho(c_f^0, p_f^0) g \omega_3 + \pi \tag{6.11}$$

for some $\pi(x)$ independent of y. Similarly, using (6.10a), the ϵ^{-1} terms of (6.2a) and the ϵ^0 terms of (6.2d) become

$$-\nabla_y \cdot \big[\rho(c_f^0, p_f^0) D_f^* \nabla_y c_f^1\big] = 0, \quad y \in \mathcal{Q}_f, \tag{6.12a}$$

$$\rho(c_f^0, p_f^0) D_f^* \big(\nabla_y c_f^1 + \nabla_x c_f^0\big) \cdot \nu = 0, \quad y \in \partial \mathcal{Q}_m. \tag{6.12b}$$

We need to find $\hat{\omega}_j(y)$, $j = 1,2,3$, which is periodic across $\partial \mathcal{Q}$ and satisfies

$$-\nabla_y \cdot [\nabla_y \hat{\omega}_j] = 0, \quad y \in \mathcal{Q}_f, \tag{6.13a}$$

$$\nabla_y \hat{\omega}_j \cdot \nu = -e_j \cdot \nu, \quad y \in \partial \mathcal{Q}_m. \tag{6.13b}$$

Then

$$c_f^1 = \sum_{j=1}^{3} \frac{\partial c_f^0}{\partial x_j}\hat{\omega}_j + \xi \tag{6.14}$$

for some $\xi(x)$ independent of y.

Finally, the ϵ^0 terms of (6.2a) and the ϵ^1 terms of (6.2d)–(6.2e) give, again using the two variable Taylor's Theorem for an analogue of (4.3),

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\phi_f^*(p_f^0)\rho(c_f^0,p_f^0)c_f^0\right] \\ &\quad + \nabla_y \cdot \left[\rho(c_f^0,p_f^0)\left(c_f^0 u_f^{*,1} + c_f^1 u_f^{*,0} - D_f^*(\nabla_y c_f^2 + \nabla_x c_f^1)\right)\right. \\ &\qquad \left. + \rho^1\left(c_f^0 u_f^{*,0} - D_f^*(\nabla_y c_f^1 + \nabla_x c_f^0)\right)\right] \\ &\quad + \nabla_x \cdot \left[\rho(c_f^0,p_f^0)\left(c_f^0 u_f^{*,0} - D_f^*(\nabla_y c_f^1 + \nabla_x c_f^0)\right)\right] \\ &\quad = 0, \quad x \in \Omega,\ y \in \mathcal{Q}_f, \end{aligned} \tag{6.15a}$$

$$\begin{aligned} &\left[\rho(c_f^0,p_f^0)\left(c_f^0 u_f^{*,1} + c_f^1 u_f^{*,0} - D_f^*(\nabla_y c_f^2 + \nabla_x c_f^1)\right)\right. \\ &\qquad \left. + \rho^1\left(c_f^0 u_f^{*,0} - D_f^*(\nabla_y c_f^1 + \nabla_x c_f^0)\right)\right] \cdot \nu \\ &\quad = \rho(c_m^0,p_m^0)\left(c_m^0 u_m^0 - D_m(u_m^0)\nabla_y c_m^0\right) \cdot \nu, \quad x \in \Omega,\ y \in \partial\mathcal{Q}_m. \end{aligned} \tag{6.15b}$$

This equation is similar to (4.14), so we argue in analogy to (4.15)–(4.17). Integrate over $y \in \mathcal{Q}_f$ and divide by $|\mathcal{Q}|$. As in (4.15), with (6.6a), we conclude that

$$\begin{aligned} &-\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f} \nabla_y \cdot \left[\rho(c_f^0,p_f^0)\left(c_f^0 u_f^{*,1} + c_f^1 u_f^{*,0} - D_f^*(\nabla_y c_f^2 + \nabla_x c_f^1)\right)\right. \\ &\qquad \left. + \rho^1\left(c_f^0 u_f^{*,0} - D_f^*(\nabla_y c_f^1 + \nabla_x c_f^0)\right)\right] dy \\ &\quad = \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_m} \frac{\partial}{\partial t}\left[\phi_m(p_m^0)\rho(c_m^0,p_m^0)c_m^0\right] dy. \end{aligned} \tag{6.16}$$

With (6.8b) and (6.11), we define the macroscopic fracture Darcy velocity as

$$\begin{aligned} u_f^0 &= \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f} u_f^{*,0}\, dy \\ &= -\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f} \mu(c_f^0)^{-1} k_f^*\Big(\sum_{j=1}^{3}\frac{\partial p_f^0}{\partial x_j}\nabla_y\omega_j - \rho(c_f^0,p_f^0)g\nabla_y\omega_3 \\ &\qquad + \nabla_x p_f^0 - \rho(c_f^0,p_f^0)ge_3\Big)dy \\ &= -\mu(c_f^0)^{-1}\sum_{j=1}^{3} e_i\Big[\sum_{k=1}^{3}\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f} k_{f,ik}^*\Big(\frac{\partial\omega_j}{\partial y_k} - \delta_{kj}\Big)dy\Big] \\ &\qquad \times\Big(\frac{\partial p_f^0}{\partial x_j} - \rho(c_f^0,p_f^0)g\delta_{j3}\Big). \end{aligned} \tag{6.17}$$

The expression in square brackets is k_f, so this is (6.5c). Now (6.14) gives us that

$$(6.18)\qquad -\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f}\nabla_x\cdot\left[\rho(c_f^0,p_f^0)(c_f^0u_f^{*,0}-D_f^*(\nabla_y c_f^1+\nabla_x c_f^0))\right]dy$$
$$=-\nabla_x\cdot\left[\rho(c_f^0,p_f^0)(c_f^0u_f^0-D_f(\nabla_x c_f^0))\right],$$

where the tensor D_f is defined by

$$(6.19)\qquad D_{f,ij}=\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f}D_f^*\left(\frac{\partial\hat{\omega}_j}{\partial y_i}+\delta_{ij}\right)dy.$$

Combining, we obtain (6.5a). We obtain (6.5b) in an entirely similar manner.

The macroscopic model is now complete. It would seem that we should replace D_f by a velocity dependent coefficient, though our homogenization does not predict this.

Appendix. Two-phase reference potentials.

In this appendix we consider the solvability of (5.4) and (5.7) for the reference potentials. For simplicity of notation, we concentrate on (5.7), which we rewrite for fixed $\Phi_{i,f}^0$ and x_3 as

$$(A.1)\qquad F_i(\Phi_{\mathrm{ref},1}^0,\Phi_{\mathrm{ref},2}^0)=0,\quad i=1,2,$$

where

$$(A.2a)\qquad F_i(\xi_1,\xi_2)=\frac{1}{|\mathcal{Q}_m|}\int_{\mathcal{Q}_m}\hat{\phi}_m(y_3-\xi_1)\hat{\rho}_i(y_3-\xi_i)\times$$
$$\times S_i(\hat{\psi}_1(y_3-\xi_1)-\hat{\psi}_2(y_3-\xi_2))\,dy$$
$$-\hat{\phi}_m(0)\hat{\rho}_i(0)S_i(\hat{\psi}_1(0)-\hat{\psi}_2(0)),$$

$$(A.2b)\qquad \hat{\psi}_i(\xi)=\psi_i(\Phi_{i,f}^0+x_3+\xi),$$

$$(A.2c)\qquad \hat{\phi}_m(\xi)=\phi_m(\hat{\psi}_1(\xi))\quad\text{and}\quad\hat{\rho}_i(\xi)=\rho_i(\hat{\psi}_i(\xi)),$$

$$(A.2d)\qquad S_1(\xi)=p_{c,m}^{-1}(-\xi)\quad\text{and}\quad S_2=1-S_1.$$

Let us clarify the mathematical assumptions on our physical quantities. We assume monotonicity: $0<\hat{\psi}_i'<\infty$, $0\le\hat{\phi}_m'<\infty$, $0\le\hat{\rho}_i'<\infty$, and $0\le\min S_1'\le S_1'=-S_2'\le\infty$. The latter assumption says that $p_{c,m}^{-1}$ has a positive minimal slope, and that its slope can become infinite at some points. Also, for some residual saturations $S_{i,r}$ and $i=1,2$,

$$0<\hat{\phi}_m(-\infty)\le\hat{\phi}_m(\xi)<1,\quad 0<\hat{\rho}_i(-\infty)\le\hat{\rho}_i(\xi)\le\hat{\rho}_i(+\infty)<+\infty,$$
$$\text{and}\quad 0\le S_{i,r}\le S_i(\xi)\le 1-S_{2/i,r}\le 1.$$

Denote the partial derivative of F_i with respect to ξ_j by $\mathcal{D}_j F_i$. These are easily computed if we note that $\hat{\psi}_i' = g\hat{\rho}_i$ by (5.1), and we define $w = \hat{\phi}_m S_1' g \geq \hat{\phi}_m(-\infty) \min S_1' \, g > 0$. Omitting the arguments,

$$\text{(A.3a)} \qquad \mathcal{D}_i F_i = \frac{-1}{|\mathcal{Q}_m|} \int_{\mathcal{Q}_m} \{(\mathcal{D}_i \hat{\phi}_m \hat{\rho}_i + \hat{\phi}_m \hat{\rho}_i')S_i(\hat{\psi}_1 - \hat{\psi}_2) + (\hat{\rho}_i)^2 w\} dy < 0,$$

$$\text{(A.3b)} \qquad \mathcal{D}_2 F_1 = \frac{1}{|\mathcal{Q}_m|} \int_{\mathcal{Q}_m} \hat{\rho}_1 \hat{\rho}_2 w \, dy > 0,$$

$$\text{(A.3c)} \qquad \mathcal{D}_1 F_2 = \frac{1}{|\mathcal{Q}_m|} \int_{\mathcal{Q}_m} \{ - \hat{\phi}_m' \hat{\rho}_2 S_2(\hat{\psi}_1 - \hat{\psi}_2) + \hat{\rho}_1 \hat{\rho}_2 w\} dy.$$

Only the sign of $\mathcal{D}_1 F_2$ is unclear. We consider the solvability of (A.1) in several distinct cases.

If the rock and fluids are incompressible, then $F_1 = -F_2$ and (A.1) gives a single condition for the saturations. By (5.1) these depend only on the difference $\Phi^0_{\text{ref},2} - \Phi^0_{\text{ref},1}$, which is uniquely defined by the strict monotonicity of S_1.

Now assume that at least one of the two fluids or the rock is compressible. Consider first the case wherein we have a single phase in the unscaled problem; that is, one of the $S_i(\hat{\psi}_1(0) - \hat{\psi}_2(0))$ is at its residual value $S_{i,r}$. If, say, the second phase is absent near $\partial\mathcal{Q}_m$, assume this to be true of the scaled problem as well, so that $S_1 = (1 - S_{2,r})$ is maximal and $S_2 = S_{2,r}$ is residual wherever it appears in (A.1)–(A.2). Then no boundary condition is needed for the second phase, $\Phi^0_{\text{ref},2}$ is not needed, and we can omit $F_2 = 0$. We get $\Phi^0_{\text{ref},1}$ from $F_1 = 0$ if the rock or first fluid are compressible; otherwise, $\Phi^0_{\text{ref},1}$ is immaterial. Similar considerations apply if the first phase is absent near $\partial\mathcal{Q}_m$.

The multiphase compressible case is the most difficult. In this case, $S_{i,r} < S_i(\hat{\psi}_1(0) - \hat{\psi}_2(0)) < 1 - S_{2/i,r}$ for $i = 1, 2$. Let us fix ξ_2 and consider whether we can solve $F_1(\xi_1, \xi_2) = 0$. There must be at least one solution ξ_1, since

$$\lim_{\xi_1 \to -\infty} F_1(\xi_1, \xi_2)$$
$$= \hat{\phi}_m(+\infty)\hat{\rho}_1(+\infty)(1 - S_{2,r}) - \hat{\phi}_m(0)\hat{\rho}_1(0)S_1(\hat{\psi}_1(0) - \hat{\psi}_2(0)) > 0$$

and

$$\lim_{\xi_1 \to +\infty} F_1(\xi_1, \xi_2) = \hat{\phi}_m(-\infty)\hat{\rho}_1(-\infty)S_{1,r} - \hat{\phi}_m(0)\hat{\rho}_1(0)S_1(\hat{\psi}_1(0) - \hat{\psi}_2(0)) < 0.$$

Since F_1 strictly decreases with ξ_1, this solution is unique, and we can define a function $\xi_1(\xi_2)$ such that $F_1(\xi_1(\xi_2), \xi_2) = 0$.

We note that

$$0 = \frac{dF_1}{d\xi_2}(\xi_1(\xi_2), \xi_2) = \mathcal{D}_1 F_1(\xi_1(\xi_2), \xi_2)\xi_1'(\xi_2) + \mathcal{D}_2 F_1(\xi_1(\xi_2), \xi_2),$$

so that

$$\text{(A.4)} \qquad \xi_1'(\xi_2) = -\frac{\mathcal{D}_2 F_1(\xi_1(\xi_2), \xi_2)}{\mathcal{D}_1 F_1(\xi_1(\xi_2), \xi_2)} > 0.$$

In fact, from (A.3),

$$\xi_1'(\xi_2) \geq \frac{\hat{\rho}_1(-\infty)\hat{\rho}_2(-\infty)\int_{\mathcal{Q}_m} w\,dy}{|\mathcal{Q}_m|\big(\max\hat{\phi}_m'\,\hat{\rho}_1(+\infty)+\hat{\phi}_m(+\infty)\max\hat{\rho}_1'\big)(1-S_{2,r})+(\hat{\rho}_1(+\infty))^2\int_{\mathcal{Q}_m} w\,dy}.$$

Since w has a positive minimum, so does $\xi_1'(\xi_2)$; consequently, $\lim_{\xi_2\to\pm\infty}\xi_1(\xi_2) = \pm\infty$.

Now consider whether we can solve $F_2(\xi_1(\xi_2),\xi_2) = 0$. First let

$$\mathcal{L} = \lim_{\xi_2\to-\infty}\frac{1}{|\mathcal{Q}_m|}\int_{\mathcal{Q}_m} S_1(\hat{\psi}_1(y_3-\xi_1(\xi_2))-\hat{\psi}_2(y_3-\xi_2))\,dy$$

and note that

$$\begin{aligned}0 &= \lim_{\xi_2\to-\infty} F_1(\xi_1(\xi_2),\xi_2)\\ &= \hat{\phi}_m(+\infty)\hat{\rho}_1(+\infty)\mathcal{L} - \hat{\phi}_m(0)\hat{\rho}_1(0)S_1(\hat{\psi}_1(0)-\hat{\psi}_2(0)),\end{aligned}$$

which implies that

$$\mathcal{L} = \frac{\hat{\phi}_m(0)\hat{\rho}_1(0)S_1(\hat{\psi}_1(0)-\hat{\psi}_2(0))}{\hat{\phi}_m(+\infty)\hat{\rho}_1(+\infty)}.$$

We use this result to show that

$$\begin{aligned}&\lim_{\xi_2\to-\infty} F_2(\xi_1(\xi_2),\xi_2)\\ &\quad= \hat{\phi}_m(+\infty)\hat{\rho}_2(+\infty)(1-\mathcal{L}) - \hat{\phi}_m(0)\hat{\rho}_2(0)S_2(\hat{\psi}_1(0)-\hat{\psi}_2(0))\\ &\quad= \big[\hat{\phi}_m(+\infty)-\hat{\phi}_m(0)\big]\hat{\rho}_2(+\infty)\\ &\qquad+\hat{\phi}_m(0)\hat{\rho}_2(+\infty)\bigg[1-\frac{\hat{\rho}_2(0)}{\hat{\rho}_2(+\infty)}S_2(\hat{\psi}_1(0)-\hat{\psi}_2(0))\\ &\qquad\qquad-\frac{\hat{\rho}_1(0)}{\hat{\rho}_1(+\infty)}S_1(\hat{\psi}_1(0)-\hat{\psi}_2(0))\bigg]\\ &\quad> 0,\end{aligned}$$

since at least one of $\hat{\phi}_m$, $\hat{\rho}_1$, and $\hat{\rho}_2$ is compressible. In a similar way, we show that $\lim_{\xi_2\to+\infty} F_2(\xi_1(\xi_2),\xi_2) < 0$. This insures that there is at least one solution to $F_2(\xi_1(\xi_2),\xi_2) = 0$. We claim that $F_2(\xi_1(\xi_2),\xi_2)$ is decreasing in ξ_2, so that the solution is unique. This will complete our demonstration of the solvability of (A.1).

Note that, with (A.4),

$$\begin{aligned}\frac{dF_2}{d\xi_2}(\xi_1(\xi_2),\xi_2) &= \mathcal{D}_1F_2(\xi_1(\xi_2),\xi_2)\xi_1'(\xi_2) + \mathcal{D}_2F_2(\xi_1(\xi_2),\xi_2)\\ &= \left(\frac{\mathcal{D}_1F_1\,\mathcal{D}_2F_2-\mathcal{D}_1F_2\,\mathcal{D}_2F_1}{\mathcal{D}_1F_1}\right)(\xi_1(\xi_2),\xi_2).\end{aligned}$$

Since the denominator is negative, we consider only the sign of the numerator. Using (A.3) and collecting expressions containing the rock and fluid compressibility (derivative) terms into $T > 0$, we see that

$$\begin{aligned}
&\mathcal{D}_1 F_1 \, \mathcal{D}_2 F_2 - \mathcal{D}_1 F_2 \, \mathcal{D}_2 F_1 \\
&\quad = T + \frac{1}{|\mathcal{Q}_m|} \int_{\mathcal{Q}_m} (\hat{\rho}_1)^2 w \, dy \, \frac{1}{|\mathcal{Q}_m|} \int_{\mathcal{Q}_m} (\hat{\rho}_2)^2 w \, dy - \left(\frac{1}{|\mathcal{Q}_m|} \int_{\mathcal{Q}_m} \hat{\rho}_1 \hat{\rho}_2 w \, dy \right)^2 \\
&\quad \geq T > 0,
\end{aligned}$$

by the Cauchy-Schwarz inequality, and our result follows.

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