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A CHARACTERIZATION OF THE SET OF ASYMPTOTIC VALUES OF A FUNCTION HOLOMORPHIC IN THE UNIT DISC

by

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ABSTRACT

It is known that the set of asymptotic values of a function meromorphic in $\{|z| < 1\}$ is characterized as being an analytic set in the extended plane. Easy examples show that there exist analytic sets in the plane which cannot be the set of asymptotic values of any function <u>holomorphic</u> in $\{|z| < 1\}$. The main object of this thesis is to obtain a characterization of the set of asymptotic values of a function holomorphic in $\{|z| < 1\}$. As a side result, it is observed that this characterization applies also to the class of normal holomorphic functions. A very simple characterization is obtained for the asymptotic set of a bounded holomorphic function having radial limits of modulus one a.e. An example is given of a bounded function holomorphic in $\{|z| < 1\}$ whose set of asymptotic values is totally disconnected.

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TO MY WIFE

JOAN

CONTENTS

1. Introduction. A function f defined in $\{|z| < 1\}$ takes the value α as an <u>asymptotic value</u> provided there exists a non-compact arc in $\{|z| < 1\}$, say $z = \varphi(t)$ for $t \in [0,1)$, with both conditions

$$\lim_{t\to 1} |\varphi(t)| = 1 ,$$

$$\lim_{t\to 1} f(\varphi(t)) = \alpha$$

satisfied. The main object of this thesis is to give a characterization of the set of asymptotic values, or briefly, the <u>asymptotic set</u>, of functions holomorphic in $\{|z| < 1\}$. We will employ the notation $\mathfrak{a}(f)$ for the asymptotic set of a function f (which may or may not be defined in $\{|z| < 1\}$).

It is well known [8,p.7] that $\mathfrak{a}(f) \neq \emptyset$ if f is holomorphic in $\{|z| < 1\}$. On the other hand, Lehto and Virtanen [5,p.58] give an example of a function f meromorphic in $\{|z| < 1\}$ with $\mathfrak{a}(f) = \emptyset$. In 1931 Mazurkiewicz [9] showed (in effect) that the asymptotic set of a function meromorphic in $\{|z| < 1\}$ is an analytic set. (The definition and useful properties of analytic sets will be found early in section 3.) In 1936 Kierst [4] constructed a function meromorphic in $\{|z| < 1\}$ whose asymptotic set was a pre-assigned analytic set in the extended plane. Thus, the asymptotic sets of functions <u>meromorphic</u> in $\{|z| < 1\}$ are characterized as being analytic sets. As a corollary, Kierst observed that, given any <u>finite</u> analytic set $\mathfrak{a}$, there exists a <u>holomorphic</u> function in $\{|z| < 1\}$ whose <u>finite</u> asymptotic values are exactly the set $\mathfrak{a}$. However, Kierst's holomorphic function necessarily has ∞ as an asymptotic value. Obviously there are functions holomorphic in $\{|z| < 1\}$ whose asymptotic sets do not contain the point ∞ (e.g., the bounded holomorphic functions). To go a step further, it is easily shown that

(1.1) <u>there exists an analytic set which cannot be the asymptotic set of any function holomorphic in</u> $\{|z| < 1\}$.

Consider, for example, the analytic set $\mathfrak{a} = [0,1]$. If $\mathfrak{a}$ were the asymptotic set of some function f holomorphic in $\{|z| < 1\}$, then the Riemann surface $\mathfrak{F}$ of f would cover points not in the interval $[0,1]$. Let b be such a point, and let R be a ray from b to ∞ which does not meet $[0,1]$. By a well known argument [8,p.7] some point of R, possibly ∞, must be an asymptotic value of f ; since $\mathfrak{a} \cap R = \emptyset$, this is a contradiction. On the other hand, $\mathfrak{a}(f)$ can be quite complicated for functions f holomorphic in $\{|z| < 1\}$: In Example 2, section 5, we obtain a bounded holomorphic function whose asymptotic set is totally disconnected. Thus we are led to the main objective of this thesis: To provide a necessary and sufficient condition under which an analytic set is the asymptotic set of some function holomorphic in $\{|z| < 1\}$. Theorem 7 provides this characterization.

In section 2 we summarize the notational conventions employed in this thesis.

Section 3 collects some of the definitions and facts needed later. In particular, analytic sets are discussed, as are the classes $\mathcal{A}$, $\mathcal{N}$ of MacLane [8], and $\mathcal{U}$ of Seidel [12,p.32].

Some preliminary results are proved in section 4. Theorem 1 establishes a fact about $f \in \mathcal{A}$ which is needed in the proof of the main theorem, Theorem 7, and which is of interest in itself. Theorem 2 shows that Theorem 7 also characterizes the asymptotic sets of $f \in \mathcal{N}$. Theorem 3 gives a Riemann surface type proof of the following revamped version of Kierst's Theorem: If $\mathfrak{a}$ is an analytic set containing ∞, then there exists $f \in \mathcal{N}$ with $\mathfrak{a}(f) = \mathfrak{a}$.

The problem of characterizing the asymptotic sets of holomorphic functions is examined in section 5. Some necessary properties of the characterization are established, with examples to show they are not also sufficient. Example 2 shows the existence of a bounded holomorphic function with totally disconnected asymptotic set. Theorem 5 gives a simple characterization of $\mathfrak{a}(f)$ for the special case $f \in \mathcal{U}$.

The main result, Theorem 7, is proved in section 6 with the help of a preliminary result, Theorem 6.

2. Notation. Most of the notation is self-evident. If f is a function, then $\check{f}$ will denote its inverse. Frequently, when we are dealing with complex-valued functions f holomorphic in $\{|z| < 1\}$, with an associated Riemann surface $\mathfrak{F}$ covering the w-plane, we shall permit ourselves the ambiguity of saying that f maps $\{|z| < 1\}$ onto $\mathfrak{F}$, and that $\check{f}$ maps $\mathfrak{F}$ onto $\{|z| < 1\}$.

∂ indicates the boundary operator, $\bar{\ }$ the closure operator, and $'$ complementation.

Completion of a proof will be indicated by the symbol $\blacksquare$.

3. Definitions and Facts. Comprehensive treatments of analytic sets can be found in [2], [7], and [14]. Following Sierpinski [14] we restrict our attention to analytic sets contained in some metric space.

(3.1) A set $\mathfrak{a}$ is <u>analytic</u> provided there exists a countable collection of closed sets $F(n_1 \ldots n_k)$, one for each finite combination $(n_1 \ldots n_k)$ of integers, and

$$\mathfrak{a} = \bigcup F_N ,$$

the union extending over all sequences $N = (n_1 n_2 n_3 \ldots)$ of integers with

$$F_N = F(n_1) \cap F(n_1 n_2) \cap F(n_1 n_2 n_3) \cap \ldots .$$

(3.2) Analytic sets were introduced by Souslin in order to characterize Borel sets; Souslin's Theorem states that a subset B of a separable, complete metric space is a Borel set if, and only if, both B and B$'$ are analytic. Thus the class of analytic sets contains all Borel sets, and is bigger. It may be shown that the class of analytic sets is closed under countable unions and intersections.

(3.3) Lusin [7,p.151] proved that bounded analytic subsets of the real line are measureable.

(3.4) A fact which will be needed later is that a continuous image of an analytic set contained in a separable complete metric space is an analytic set [14,p.219]. However, the property of analytic sets which makes them so use- for our purposes is given in the following form of

(3.5) THEOREM OF LUSIN AND SIERPINSKI. *A subset* $\mathfrak{a}$ *of a separable, complete metric space is analytic if, and only if, there exists a function* α *from* $(-\infty,\infty)$ *onto* $\mathfrak{a}$ *which is continuous on the left* [14,p.221]. That is, $\alpha(x-) = \alpha(x)$. By a straightforward modification we may assume that α is defined only on $(0,1]$.

(3.6) Facts about universal covering surfaces are well known and can be obtained, for instance, in [1]. However, we wish to emphasize a useful fact which is employed several times in the proofs to follow. To get oriented, let $\mathfrak{F}$ be a surface covering the w-plane; an *asymptotic path* on $\mathfrak{F}$ is a non-compact arc Γ in $\mathfrak{F}$ whose projection into the w-plane is an arc which ends at a *point*, called the *asymptotic value determined by* Γ. We denote by $\mathfrak{a}(\mathfrak{F})$ the set of asymptotic values determined by asymptotic paths on $\mathfrak{F}$. Then: *If* $\tilde{\mathfrak{F}}$ *is the universal covering surface of* $\mathfrak{F}$, *then* $\mathfrak{a}(\tilde{\mathfrak{F}}) = \mathfrak{a}(\mathfrak{F})$. The proof of this fact is straightforward, making use of the fact that there exists a local homeomorphism of $\tilde{\mathfrak{F}}$ onto $\mathfrak{F}$, and will be omitted here.

(3.7) A function f defined in $\{|z| < 1\}$ is said to have a *point asymptotic value* α at ζ, $|\zeta| = 1$, if there is an arc $\Gamma \subset \{|z| < 1\}$ tending to ζ on which f tends to the value α. If S is an arbitrary set of complex numbers, then we denote by $A(S)$ the set of points on $\{|z| = 1\}$ at each of which f has a point asymptotic value in the set S. Now let $\mathfrak{w}$ denote the extended complex plane. $f \in \mathcal{A}$ if and only if f is holomorphic non-constant in $\{|z| < 1\}$

and $A(\mathfrak{b})$ is dense on $\{|z| = 1\}$ [8,p.8].

MacLane introduced the class $\mathcal{A}$ in [8] and proved there the following two facts which we shall use later.

(3.9) Let $f \in \mathcal{A}$ and let $\{\gamma_n\}$ be a sequence of simple arcs in $\{|z| < 1\}$ which tend to an arc γ of $\{|z| = 1\}$. If

$$\mu_n = \max_{z \in \gamma_n} |f(z)|$$

then $\mu = \liminf \mu_n > 0$ [8,Theorem 9].

(3.10) Let $f \in \mathcal{A}$ and let γ_n be a sequence of distinct simple arcs in $\{|z| < 1\}$ which tend to the arc γ of $\{|z| = 1\}$. If

$$\inf_{z \in \gamma_n} |f(z)| = \mu_n$$

and $\mu_n \to \infty$ as $n \to \infty$, then there exists an arc in $\{|z| < 1\}$ tending to γ on which f has the asymptotic value ∞ [8,Theorem 3].

(3.11) The class $\mathcal{N}$ of <u>normal</u> <u>holomorphic</u> <u>functions</u> is defined as follows [8,p.43]: $f \in \mathcal{N}$ if f is holomorphic, non-constant in $\{|z| < 1\}$ and the set of functions $f(T(z))$, where T ranges over all the linear transformations preserving $\{|z| < 1\}$, is a normal family. Thus, $f \in \mathcal{N}$ if f is holomorphic, non-constant in $\{|z| < 1\}$ and omits two finite values.

(3.12) MacLane showed [8,Theorem 17] that $\mathcal{N} \subset \mathcal{A}$ and that if $f \in \mathcal{N}$, then f has only point asymptotic values. Moreover, if $f \in \mathcal{N}$, and g is bounded, holomorphic, then $f + g \in \mathcal{N}$ [5,p.53].

(3.13) $f \in \mathcal{U}$ if f is holomorphic, non-constant, bounded in $\{|z| < 1\}$ and has radial limits of modulus one a.e. [12,p.32]. It is known [12,p.36] that if $f \in \mathcal{U}$, then $\{|w| = 1\} \subset \mathfrak{C}(f)$.

(3.14) An <u>interior</u> function is a map which carries open sets into open sets and is non-constant on non-degenerate continua. Stoïlow's theorem [15,p.237] gives the following result: If f is an interior map from $\{|\zeta| < 1\}$ into the finite plane, then there exists a homeomorphism h of $\{|z| < 1\}$ onto $\{|\zeta| < 1\}$ such that $f \circ h$ is holomorphic in $\{|z| < 1\}$.

4. Preliminary Results. If $f \in \mathcal{A}$ and B is a Borel subset of the sphere, then MacLane [8, Theorem 10] proved that $A(B)$ is measurable. McMillan [10] generalized this result: If B is a Borel set on the sphere, and f is holomorphic in $\{|z| < 1\}$, then $A(B)$ is a Borel set. The following theorem extends these results to the class of analytic subsets of the sphere, and Corollary 1 provides a fact used in the proof of Theorem 7.

THEOREM 1. *Let $f \in \mathcal{A}$ have only point asymptotic values. If $\mathfrak{a}$ is an analytic subset of the sphere, then $A(\mathfrak{a})$ is an analytic subset of $\{|z| = 1\}$.*

Proof. *Preliminaries.* Let $\mathfrak{F}$ be the Riemann surface of f covering the w-plane. If π is the projection map of $\mathfrak{F}$ into the w-plane, then $\mathfrak{F}$ becomes a metric space if we define a metric by

$$\rho(P_1,P_2) = \inf_{\Gamma} \text{diam}\, \pi(\Gamma),$$

where Γ is an arc on $\mathfrak{F}$ from P_1 to P_2 and the diameter of $\pi(\Gamma)$ is measured in the spherical metric of the extended w-plane.

Let $\mathfrak{F}^*$ be the *completion* of $\mathfrak{F}$; that is, points of $\mathfrak{F}^*$ are equivalence classes of Cauchy sequences of points of $\mathfrak{F}$. (Two Cauchy sequences, $(P_1,P_2,\cdots)$ and $(Q_1,Q_2,\cdots)$ are equivalent if $\lim \rho(P_n,Q_n) = 0$.) If we identify the Cauchy sequence $(P,P,\cdots)$ with $P \in \mathfrak{F}$, then $\mathfrak{F} \subset \mathfrak{F}^*$. If $P^* = (P_1,P_2,\cdots)$ and $Q^* = (Q_1,Q_2,\cdots)$ are two points of

$\mathfrak{F}^*$, then we define a metric on $\mathfrak{F}^*$ by

$$\rho(P^*,Q^*) = \lim_{n\to\infty} \rho(P_n,Q_n),$$

and we are permitted the ambiguity of notation since the metric on $\mathfrak{F}^*$ coincides on $\mathfrak{F}$ with the previously defined metric. Finally, it is evident from the separability of $\mathfrak{F}$ that $\mathfrak{F}^*$ is a separable, complete metric space.

We want now to extend the projection map π to the space $\mathfrak{F}^*$. If $P^* = (P_1,P_2,\cdots)$ is a point of $\mathfrak{F}^*$ then $(\pi(P_1),\pi(P_2),\cdots)$ is a Cauchy sequence with respect to the spherical metric, and $\lim \pi(P_n) = p$ exists as a point in the extended w-plane. We define

$$\pi(P^*) = p.$$

Note that $\pi(\mathfrak{F}^*-\mathfrak{F}) = \mathfrak{A}(f)$, the asymptotic set of f. For, if $P^* = (P_1,P_2,\cdots) \in \mathfrak{F}^*-\mathfrak{F}$, then we may join P_n to P_{n+1} by an arc Γ_n on $\mathfrak{F}$ such that diam $\pi(\Gamma_n)$ tends to zero. Then, if Γ is the arc on $\mathfrak{F}$ obtained by joining the Γ_n together, $\pi(\Gamma)$ is an arc ending at the point $\pi(P^*) = p$. Thus Γ corresponds to an arc in $\{|z| < 1\}$ on which f has the point asymptotic value p. On the other hand, if $p \in \mathfrak{A}(f)$, then there is an arc $\Gamma \subset \mathfrak{F}$ corresponding to a non-compact arc in $\{|z| < 1\}$ on which f has the point asymptotic value p. Let $(p_1,p_2,\cdots)$ be a Cauchy sequence (with respect to the spherical metric) of points on $\pi(\Gamma)$ with limit point p. Let P_n be the point of Γ covering p_n; then $P^* = (P_1,P_2,\cdots)$ has the properties $P^* \in \mathfrak{F}^*$ and $\pi(P^*) = p$.

Moreover, π is a continuous map of $\mathfrak{S}^*$ into the extended plane: Let $\pi(P^*) = p$ and $\epsilon>0$. Then

$$\{Q^* \in \mathfrak{S}^*: \quad \rho(P^*,Q^*)<\epsilon\}$$

is open and is mapped by π into

$$\{q \in \mathfrak{W}: \sigma(q,p)<\epsilon\}$$

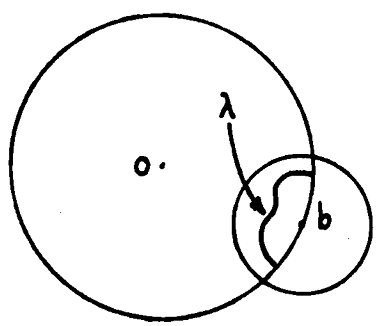
where σ is the spherical metric of the extended w-plane $\mathfrak{W}$. Thus, π is continuous.

Next, we extend the homeomorphism $\check{f}$ (see section 2) from $\mathfrak{S}$ to $\{|z|<1\}$ to a continuous mapping of $\mathfrak{S}^*$ into $\{|z|\le 1\}$. Let $P^* \in \mathfrak{S}^* - \mathfrak{S}$; there exists an asymptotic path Γ in $\mathfrak{S}$ tending to P^* which determines the asymptotic value $\pi(P^*) \in \mathfrak{A}(f)$. The image of Γ under $\check{f}$ is an arc γ in $\{|z|<1\}$ tending to a <u>point</u> b of $\{|z|=1\}$ (since f has only point asymptotic values). Define

$$\check{f}(P^*) = b.$$

To show that $\check{f}$ is well-defined on $\mathfrak{S}^* - \mathfrak{S}$, suppose Γ_1 and Γ_2 are two arcs on $\mathfrak{S}$ tending to P^*. Then there exists sequences of points $P_n \in \Gamma_1$, $Q_n \in \Gamma_2$ with $\lim P_n = \lim Q_n = P^*$ such that P_n and Q_n can be joined by an arc $\Lambda_n \subset \mathfrak{S}$ for which $\pi(\Lambda_n)$ is arbitrarily close to $\pi(P^*)$ (in the spherical metric), provided n is large enough. Let $\check{f}(\Lambda_n) = \lambda_n$, an arc in $\{|z|<1\}$; if $\gamma_1 = \check{f}(\Gamma_1)$ and $\gamma_2 = \check{f}(\Gamma_2)$ end at <u>distinct</u> points of $\{|z|=1\}$, then the λ_n tend to an arc of $\{|z|=1\}$. If $\pi(P^*) \neq \infty$, then we obtain a contradiction to the fact that the function $f(z) - \pi(P^*) \in \mathcal{A}$, using (3.9). If $\pi(P^*) = \infty$, then we obtain a contradiction to the fact that $f(z)$ has only <u>point</u> asymptotic values, using (3.10). Thus γ_1 and γ_2 must end at the <u>same</u> point of $\{|z|=1\}$, and $\check{f}$ is accordingly well-defined.

To conclude the preliminaries, we want to show that $\check{f}$ is continuous on $\mathfrak{F}^*$. It suffices to show that $\check{f}$ is continuous on $\mathfrak{F}^* - \mathfrak{F}$. Thus, let $P^* \in \mathfrak{F}^* - \mathfrak{F}$ with $b = \check{f}(P^*)$, and let $\varepsilon>0$. Since $f \in \mathcal{A}$, there exists (figure 1) a crosscut λ of $\{|z|<1\}$ in $\{|z-b|<\varepsilon\}$ which separates $z = 0$ and $z = b$, and whose endpoints on $\{|z|=1\}$ determine asymptotic



$|z|<1$

Figure 1.

values of f. The image $\Lambda = f(\lambda)$ in $\mathfrak{F}^*$ is <u>closed</u>, hence $\rho(\Lambda,P^*) = d>0$: Otherwise, there would exist $Q^* \in \Lambda$ such that $\rho(P^*,Q^*) = 0$, i.e., $P^* \in \Lambda$. But this would violate the fact that $b \notin \lambda$. Now the open set

$$\mathfrak{u}^* = \{Q^* \in \mathfrak{F}^*: \ \rho(P^*,Q^*)<d\}$$

is mapped by $\check{f}$ into $\{|z-b|<d\}$, since every point of $\mathfrak{u}^*$

can be joined to P^* by an arc which does not meet Λ. Thus, $\check{f}$ is continuous at P^*.

Conclusion of the proof. Let $\mathfrak{a}$ be an arbitrary analytic set in the extended w-plane. Then $\mathfrak{a}$ is of the form (3.1)

$$\mathfrak{a} = \bigcup F(n_1) \cap F(n_1 n_2) \cap \cdots$$

where the $F(n_1 \cdots n_k)$ are a countable collection of closed sets, one for each finite set of integers $(n_1 \cdots n_k)$, and the union is taken over all infinite sequences of integers. Since π is continuous, $\check{\pi}(F(n_1 \cdots n_k)) \equiv F^*(n_1 \cdots n_k)$ is closed in $\mathfrak{J}^*$. Hence

$$\mathfrak{a}^* = \check{\pi}(\mathfrak{a}) = \bigcup F^*(n_1) \cap F^*(n_1 n_2) \cap \cdots$$

is an analytic subset of $\mathfrak{J}^*$. But then, (3.4), $\check{f}(\mathfrak{a}^*)$ is an analytic subset of $\{|z| \leq 1\}$. Since, (3.2),

$$A(\mathfrak{a}) = \{|z|=1\} \cap \check{f}(\mathfrak{a}^*)$$

is the intersection of two analytic sets, the result follows. ∎

Remark. The proof of this theorem was inspired by Mazurkiewicz [9] who used this technique to prove (in effect) that $\mathfrak{a}(f)$ is analytic for f meromorphic. This result, for f holomorphic in $\{|z|<1\}$, is contained in the above proof. For, $\mathfrak{a}(f) = \pi(\mathfrak{J}^* - \mathfrak{J})$ is the continuous image of a closed subset of a separable, complete metric space, and (3.4) applies.

COROLLARY 1. *Let* h *be a conformal homeomorphism of* *a simply connected domain* D *onto* $\{|w|<1\}$. *Let* $\mathfrak{A}$ *be an analytic subset of* (accessible ∂D) *and denote by* $h(\mathfrak{A})$ *the map of* $\mathfrak{A}$ *onto* $\{|w|=1\}$ *determined by the Carathéodory boundary correspondence. Then* $h(\mathfrak{A})$ *is an analytic set.*

COROLLARY 2. $\mathfrak{A}$ *analytic implies that* $A(\mathfrak{A})$ *is measurable.*

The proof of Corollary 1 follows from the fact that $h(\mathfrak{A})$ is simply $A(\mathfrak{A})$ for $\check{h}$. Corollary 2 follows from (3.3).

The next theorem extends the results of Theorem 6 to the class of normal functions.

THEOREM 2. *If* f *is holomorphic in* $\{|z|<1\}$, *then there exists* $g \in \mathcal{N}$ *with* $\mathfrak{A}(g) = \mathfrak{A}(f)$.

Proof. Lohwater and Piranian [6, Theroem 6] have obtained the following result: If $E \subset \{|z|=1\}$ is of type F_σ and of measure zero, then there exists a function bounded and holomorphic in $\{|z|<1\}$ whose radial limits exist everywhere on $\{|z|=1\}-E$ and nowhere on E.

Now if $\mathfrak{A}(f)$ contains only countably many points, then $\infty \in \mathfrak{A}(f)$, using the argument of (1.1). Moreover, the result of adding a bounded function to a normal function is again a normal function (3.12). Hence by judiciously selecting the set E, we may add a Lohwater-Piranian function to the modular function and so obtain a normal function whose asymptotic set contains only ∞, or ∞ and one finite point. The modular function itself is an example of a normal function whose asymptotic set contains ∞ and two finite points. Thus, applying a suitable linear transformation, we obtain, in the case where $\mathcal{A}(f)$ contains three or less points, a normal function g with $\mathcal{A}(g) = \mathcal{A}(f)$.

In the case where $\mathcal{A}(f)$ contains at least two finite values, say a and b, let $\mathfrak{F}$ be the Riemann surface of f covering the w-plane. If f omits the values a and b, then $f \in \mathcal{N}$, (3.11), and we are done. Otherwise remove all points of $\mathfrak{F}$ over a and b to obtain a Riemann surface $\mathcal{G}$. The universal covering surface $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ corresponds to a function g holomorphic in $\{|z|<1\}$ which belongs to $\mathcal{N}$, since it omits three values. Moreover, from the remark (3.6), we see that $\mathcal{A}(g) = \mathcal{A}(f)$. ▮

We conclude this section by giving a constructive proof of Kierst's Theorem [4], for the case f holomorphic in $\{|z|<1\}$. Rather than appealing to Theorem 2 and Kierst's Theorem, we obtain $f \in \mathcal{N}$ directly as a feature of the construction.

THEOREM 3. *If* $\mathcal{A}$ *is an analytic set containing* ∞, *then there exists* $f \in \mathcal{N}$ *with* $\mathcal{A}(f) = \mathcal{A}$.

Proof. Just as in the proof of Theorem 2, if $\mathcal{A}$ contains less than three points we may construct the proper function from a modular function and a Lohwater-Piranian type function. Thus we may assume that a, b, and $\infty \in \mathcal{A}$. The proof consists in exhibiting a simply connected hyperbolic Riemann surface $\mathfrak{F}$ whose associated holomorphic function has the desired properties.

To begin with, the Theorem of Lusin and Sierpinski (3.5) gives us a function $\alpha : (0,1] \to \mathbb{C} - \{\infty\}$ onto and continuous on the left. For each finite sequence of indices $(i_1 \cdots i_n)$, $i_k = 0,1$, choose a point $P(i_1 \cdots i_n) \neq 0$ satisfying

(1) $|P(i_1 \cdots i_n) - \alpha(\sum_1^n i_k 2^{-k})| < 2^{-n}$,

(2) given $(i_1 \cdots i_n)$, none of the four triples $(e, P(i_1 \cdots i_n), P(i_1 \cdots i_n i))$, $e = a,b$, $i = 0,1$ are collinear,

(3) the points a,b, $P(i_1 \cdots i_n)$, $P(i_1 \cdots i_n 0)$, $P(i_1 \cdots i_n 1)$ have different arguments.

<u>Construction of the surface</u> $\mathfrak{F}$. First we need to describe the components which we will fasten together to obtain a preliminary surface $\hat{\mathfrak{F}}$.

S : The plane cut along the two rays

$$R(i) = \{|P(i)| \le |z| < \infty;\ \arg z = \operatorname{Arg} P(i)\},\quad i = 0,1.$$

S($i_1 \ldots i_n$): The plane cut along the three rays $R(i_1 \cdots i_n)$ and

$$R(i_1 \cdots i_n i) = \{|P(i_1 \cdots i_n i)| \le |z| < \infty;\ \arg z = \operatorname{Arg} P(i_1 \ldots i_n i)\},\quad i = 0,1.$$

L($i_1 \ldots i_n$): The plane cut along the ray $R(i_1 \cdots i_n)$.

We construct $\hat{\mathfrak{F}}$ as follows:

1st Level: To S adjoin along each cut $R(i_1)$ a copy of $S(i_1)$ so that first order branch points are formed at the points $P(i_1)$, $i_1 = 0,1$. The simply connected bordered surface $\mathfrak{F}_1$ so formed has "free" edges along the cuts $R(i_1 i_2)$ in $S(i_1)$.

Nth Level: Having obtained $\mathfrak{F}_{N-1}$ as a simply connected bordered surface with 2^N cuts along the rays $R(i_1 \cdots i_N)$ in $S(i_1 \cdots i_{N-1})$, we obtain $\mathfrak{F}_N$ as follows: Along the cut $R(i_1 \cdots i_N)$ we hang <u>two</u> copies of $L(i_1 \cdots i_N)$ and a copy of $S(i_1 \cdots i_N)$, so that a third order branch point is created at $P(i_1 \cdots i_N)$, according to the scheme of figure 2. Here

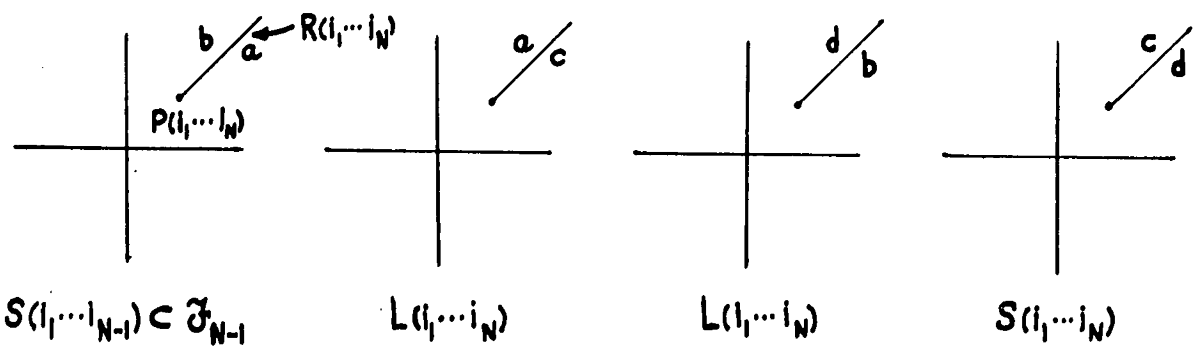


Figure 2.

the appropriate edges are identified by letter. The resulting simply connected bordered surface $\mathfrak{F}_N$ has 2^{N+1} cuts along the Rays $R(i_1 \cdots i_{N+1})$ of the sheets $S(i_1 \cdots i_N)$ just attached.

We take $\hat{\mathfrak{F}} = \bigcup \mathfrak{F}_N$; it is clear that $\hat{\mathfrak{F}}$ is a simply connected Riemann surface. At this point the easiest way to obtain $\mathfrak{F}$ would be to remove all points of $\hat{\mathfrak{F}}$ over a and b, and form the universal covering surface. However, we obtain $\mathfrak{F}$ by the following elementary construction. Let $\mathfrak{M}$ be the modular surface having no points over a, b, and ∞. Each sheet of $\hat{\mathfrak{F}}$ is either S, or a $S(i_1 \cdots i_n)$, or a $L(i_1 \ldots i_n)$. We cut each of these sheets along <u>disjoint</u> arcs from a and b to ∞, being sure that none of these cuts sever the branch lines $R(i_1 \cdots i_n)$ used in the construction of $\hat{\mathfrak{F}}$, and that the cuts made on each $S(i_1 \cdots i_n)$ do not sever the lines joining $P(i_1 \cdots i_n)$ to $P(i_1 \cdots i_n 0)$ and $P(i_1 \cdots i_n 1)$. (This is the purpose of condition (2).) To each of these fresh cuts in $\hat{\mathfrak{F}}$ we attach copies of $\mathfrak{M}$, cut along the corresponding line, so that logarithmic branch points are created at the endpoint of the cut, i.e., at either a or b. The result is a simply connected Riemann surface $\mathfrak{F}$ having no points over a, b, or ∞. For this reason, $\mathfrak{F}$ is hyperbolic and the associated holomorphic function f is normal. It remains to show that $\mathfrak{C}(f) = \mathfrak{C}$.

First, $\mathfrak{C} \subset \mathfrak{C}(f)$. By construction, $\infty \in \mathfrak{C}(f)$; if $\beta \in \mathfrak{C} - \{\infty\}$, then $\alpha(x) = \beta$ for some $x \in (0,1]$. Let

$$x = \sum_{1}^{\infty} i_k 2^{-k}, \; i_k = 0,1$$

be the non-terminating binary expansion of x. Let Γ be the polygonal path on $\mathfrak{F}$ whose successive vertices are the branch points $P(i_1)$, $P(i_1 i_2)$, $P(i_1 i_2 i_3)$, $\cdots$. The projection γ of Γ into the extended w-plane is an arc tending to β, since

$$|P(i_1 \cdots i_n) - \beta| \le |P(i_1 \cdots i_n) - \alpha(\sum_{1}^{n} i_k 2^{-k})| + |\alpha(\sum_{1}^{n} i_k 2^{-k}) - \beta| < 2^{-n} + \epsilon_n \;,$$

where $\varepsilon_n \to 0$, since α is continuous on the left. Now the image of Γ in $\{|z|<1\}$ is a non-compact arc tending to a point of $\{|z|=1\}$, since $f \in \mathcal{N}$. Thus $\beta \in \mathfrak{A}(f)$.

Finally, $\mathfrak{A}(f) \subset \mathfrak{A}$: Let $\beta \in \mathfrak{A}(f)$; as $a, b, \infty \in \mathfrak{A}$, we may assume $\beta \neq a,b,\infty$, and hence $|\beta|<\infty$. Thus, there exists a non-compact arc Γ on $\mathfrak{F}$ whose projection γ in the w-plane tends to β. Let $P = h(t)$ be the continuous map from $[0,1)$ to Γ. As $\beta \neq a,b,\infty$, Γ cannot end in some $\mathfrak{m}$ added to $\hat{\mathfrak{F}}$, or in some subsurface $\mathfrak{F}_N$ of $\hat{\mathfrak{F}}$. Hence there exists a unique monotone increasing sequence $\{t_k\}$ of values in $[0,1)$, with $\lim t_k = 1$, satisfying the condition that t_k is the largest value of t for which $h(t) \in \bar{\mathfrak{F}}_{k-1}$. Now $h(t_k)$ lies on a boundary component of $\mathfrak{F}_{k-1}$, say on the branch line $R(i_1 \cdots i_k)$, so that $h(t_k)$ uniquely determines the endpoint $P(i_1 \cdots i_k)$ of this ray. Now

(4.1) $$\lim_{k\to\infty} |h(t_k) - P(i_1 \cdots i_k)| = 0;$$

otherwise there exists $\varepsilon>0$ such that for infinitely many k,

$$|h(t_k) - P(i_1 \cdots i_k)| \geq \varepsilon.$$

However, since γ tends to β, we can find a k_0 such that for some $k_1 \geq k_0$ and $t \in [t_{k_1},1)$, we have both

$$|h(t_{k_1}) - P(i_1 \cdots i_k)| \geq \varepsilon$$

and $$|h(t) - h(t_{k_1})|<\varepsilon.$$

By the construction of $\mathfrak{F}$, this means that Γ must be confined to a finite sheeted component of $\mathfrak{F}$ over $\{|w - h(t_{k_1})|<\varepsilon\}$, contradicting the fact that Γ meets infinitely many of the branch lines $R(i_1\cdots i_n)$. This establishes (4.1) and hence

$$\beta = \alpha\Big(\sum_1^\infty i_k 2^{-k}\Big),$$

where the i_k's are determined by the $P(i_1\cdots i_k)$ associated with the $h(t_k)$, since

$$\begin{aligned}|\beta - \alpha\Big(\sum_1^\infty i_n 2^{-n}\Big)| \le |\beta - h(t_k)| &+ |h(t_k) - P(i_1\cdots i_k)| \\ &+ |P(i_1\cdots i_k) - \alpha\Big(\sum_1^k i_n 2^{-n}\Big)| \\ &+ |\alpha\Big(\sum_1^k i_n 2^{-n}\Big) - \alpha\Big(\sum_1^\infty i_n 2^{-n}\Big)|\end{aligned}$$

Thus $\mathfrak{a}(f) = \mathfrak{a}$. $\blacksquare$

Remark. The method of proof used above is similar to that of Heins [3], who extended Kierst's Theorem to the class of entire functions.

5. Necessary Conditions; the Class $\mathcal{U}$. We have already remarked, (1.1), that there exist analytic sets which are not asymptotic sets for any functions <u>holomorphic</u> in $\{|z| < 1\}$. However, we can easily establish some

Necessary Conditions on $\mathfrak{a}(f)$. Let f be holomorphic in $\{|z| < 1\}$, and put $D = f(\{|z| < 1\})$. Then

(5.1) $\mathfrak{a}(f)$ <u>is analytic</u>. This result of Mazurkiewicz [9] was obtained in the remark following Theorem 1.

(5.2) $\partial D \subset \mathfrak{a}(f)^{-} \subset D^{-}$. Let $\mathfrak{F}$ be the Riemann surface of f over the w-plane; if π is the projection map, then $\pi(\mathfrak{F}) = D$. Suppose first $b \in \partial D$; then there are points of D arbitrarily close to b. If L is a straight line segment from a point of D to b, then L must contain an asymptotic value of f (which may be b): simply consider a maximal lifting of L into $\mathfrak{F}$, beginning at the endpoint of L in D. Hence either $b \in \mathfrak{a}(f)$ or b is a limit point of points in $\mathfrak{a}(f)$, i.e., $\partial D \subset \mathfrak{a}(f)^{-}$. Next, any asymptotic value of f must have points of D arbitrarily close, since asymptotic values are determined by non-compact arcs on $\mathfrak{F}$. Hence, $\mathfrak{a}(f)^{-} \subset D^{-}$.

(5.3) <u>if</u> $b \in \partial D$ <u>is accessible from</u> D, <u>then either</u> $b \in \mathfrak{a}(f)$, or every arc in D to b must meet $\mathfrak{a}(f)$. Just as in (5.2), this result is proved by lifting arcs into $\mathfrak{F}$.

(5.4) <u>if</u> $b \in \partial D$ <u>is inaccessible from</u> D, <u>then</u> $b \notin \mathfrak{a}(f)$. For, if $b \in \mathfrak{a}(f)$, then there would be an arc Γ

on $\mathfrak{Z}$ whose projection γ tends to b. But since $\Gamma \subset \mathfrak{Z}$, $\gamma \subset D$, so that b would be accessible from D.

One naturally asks the question: Are the above necessary conditions also <u>sufficient</u>? I.e., given a domain D and an analytic set $\mathfrak{a}$ satisfying the conditions satisfied by $\mathfrak{a}(f)$ in (5.1), (5.2), (5.3), (5.4), is there a function holomorphic in $\{|z| < 1\}$ whose asymptotic set is $\mathfrak{a}$? In view of (5.3), the following example shows that (5.1), (5.2), and (5.4) are not, in themselves, sufficient:

EXAMPLE 1. *Let* $\mathfrak{a}$ *be the set* $\{|z| = 1\} - \{1\}$. In this case, $\mathfrak{a}$ satisfies (5.1), (5.2), and (5.4) with $D = \{|z|<1\}$, but by (5.3), $\mathfrak{a}$ cannot be the asymptotic set of any function holomorphic in $\{|z| < 1\}$.

It is not known at this time whether (5.3), *together with* (5.1), (5.2), *and* (5.4), *is a sufficient condition*.

The following extension of (5.2) is cast in the form of a theorem for later use.

THEOREM 4. *If* f *is holomorphic in* $\{|z| < 1\}$, *then there exists a simply connected domain* D_0 *with the properties*

(1) $\partial D_0 \subset \mathfrak{a}(f)^- \subset D_0^-$,

(2) (inaccessible from D_0) $\partial D_0 \cap \mathfrak{a}(f) = \phi$,

(3) $\infty \notin D_0$.

Proof. If $\infty \in \mathfrak{a}(f)$, simply take $D_0 = \{|z| < \infty\}$. For the case $\infty \notin \mathfrak{a}(f)$, some preliminaries are required to obtain D_0. First define

$$D = f(\{|z| < 1\}) ,$$

$$D_n = f(\{|z| < 1 - \tfrac{1}{n}) , \quad n \geq 2 ,$$

and note that $\{D_n\}$ is an increasing sequence of bounded domains in the w-plane with $\partial D_n \subset D$. Let D_n^∞ be the unbounded component of $D_n^{-\prime}$. Then $D_n^* \equiv (D_n^\infty)^{-\prime}$ is a simply connected, bounded domain with

(A) $D_n \subset D_n^*$

(B) $\partial D_n^* \subset \partial D_n$;

the first inclusion follows from the definition of D_n^∞. The second results from the fact that for any set S, both $\partial S^- \subset \partial S$ and $\partial(S^{-\prime}) = \partial(S^-)$ hold. Then (B) follows, since

$$\partial D_n^* = \partial[(D_n^\infty)^{-\prime}] = \partial(D_n^\infty)^- \subset \partial D_n^\infty ;$$

but D_n^∞ is a component of $D_n^{-\prime}$, so

$$\partial D_n^\infty \subset \partial(D_n^{-\prime}) = \partial D_n^- \subset \partial D_n .$$

As a final preliminary, we note that the D_n^* form an increasing sequence of simply connected domains. Thus

$$D_o \equiv \bigcup_2^\infty D_n^*$$

is a simply connected domain. Observe that D_n^* is just the maximal simply connected, bounded domain such that $\partial D_n^* \subset \partial D_n$. That is, D_n^* "fills in the holes in D_n."

It remains to show that D_o has the desired properties.

First, $\partial D_o \subset \mathfrak{C}(f)^- \subset D_o^-$; this follows from (5.2) if we show that $D \subset D_o$ and $\partial D_o \subset \partial D$. To obtain the first inclusion, note $D = \bigcup D_n \subset \bigcup D_n^* = D_o$. For the second inclusion, let

$b \in \partial D_o$; then $b \notin D$. But every neighborhood of b contains points of some D_n^*, hence a point of $\partial D_n^* \subset \partial D_n$ by (B). But $\partial D_n \subset D$, so every neighborhood of b contains points of D, i.e., $b \in \partial D$.

Also, (inaccessible) $\partial D_o \cap \mathfrak{a}(f) = \phi$, since inaccessible boundary points of D_o must be inaccessible boundary points of D, and (5.4) applies.

Finally, $\infty \notin D_o$, since each D_n^* is bounded. ▯

Remark. Example 1 shows that the necessary condition given by Theorem 4 is not sufficient. That is, if $\mathfrak{a}$ is an analytic set and D_o is a simply connected domain satisfying the three conditions of the theorem (with $\mathfrak{a}(f)$ replaced by $\mathfrak{a}$), then it does not follow that $\mathfrak{a}$ is the asymptotic set of some function holomorphic in $\{|z| < 1\}$.

A natural thought would be to search for topological properties which an asymptotic set must possess. However, in view of (1.1) and Example 1, questions of compactness are immediately excluded. The following example shows that questions of connectivity must also be excluded.

EXAMPLE 2. <u>There exists a bounded, holomorphic function in</u> $\{|z| < 1\}$ <u>whose asymptotic set is totally disconnected</u>.

Proof. We will obtain below a subdomain B of $\{|z|<1\}$ whose accessible boundary is a totally disconnected set. Using the universal covering surface $\tilde{B}$ of B, we obtain the desired function, in view of (3.6).

Let T be the open isosceles triangle (i.e., a simply connected region of triangular shape) in $\{y > 0\}$ of height three abutting on $[0,1]$. Let $\Gamma = \partial T \cap \{y>0\}$. The construction below proves the existence of a subdomain D of T whose accessible boundary includes Γ plus a totally disconnected set; moreover $D^- = T^-$. It remains merely to map T^- topologically onto sectors of $\{|w| \le 1\}$ to obtain the desired domain.

In order to facilitate the description of D, three types of auxiliary sets will be defined: Necks $\mathfrak{h}$, laces $\mathcal{L}$, and diamonds $\mathfrak{D}$. D will arise as the complement in T of the diamond necklace $\bigcap \mathfrak{D}$.

Necks. Let L be a closed line segment and $h > 0$. A neck $\mathfrak{h}(L,h)$ about L of width h is a set constructed as follows: Let I_n, $n=1,2,3,\ldots$ be the open intervals removed from L in the standard construction of the Cantor subset of L, with I_1 the "middle third." Let T_1 be an isosceles (closed) triangle of height h whose base is I_1. For each I_n, let T_n be a triangle congruent to T_1 whose base is I_n. Put $\mathfrak{T} = \bigcup_1^\infty T_n \cup L$, and let $\mathfrak{T}_1$ be the reflection of $\mathfrak{T}$ in L. Then

$$\mathfrak{h}(L,h) = \mathfrak{T} \cup \mathfrak{T}_1 .$$

Clearly $\mathfrak{h}$ is closed and contains the line segment L. This is illustrated by Figure 3.

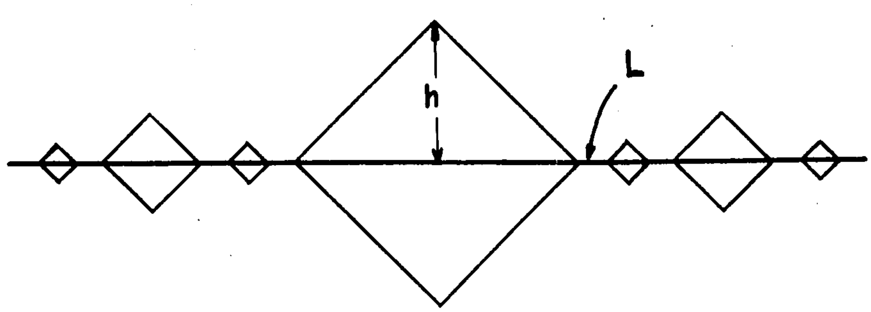


Figure 3

Laces. A lace will be a countable collection of lines parallel to $\{y = 0\}$. Intuitively, $\mathcal{L}_1$ consists of lines clustering on $\{y = 0\}$. Then $\mathcal{L}_k$ is obtained from $\mathcal{L}_{k-1}$ by first adding in a neighborhood of each line $L \in \mathcal{L}_{k-1}$ a sequence of lines which cluster on L from both sides, then by removing $\mathcal{L}_{k-1}$ from the set of lines so obtained. To make these ideas precise, a little bit of arithmetic is necessary:

0th Lace. Take $\mathcal{L}_0 = \{y = 0\}$.

1st Lace. Let n_1 be a positive integer and $e_1 = \pm 1$. Define $L(e_1 n_1) = \{y = e_1 2^{-n_1}\}$, and set

$$\mathcal{L}_1 = \bigcup_{n_1=1}^{\infty} L(\pm n_1) .$$

2nd Lace. For $i=1,2$, let n_i be positive integers and $e_i = \pm 1$. Define

$$L(e_1 n_1, e_2 n_2) = \{y = e_1 2^{-n_1} + 2^{-2} e_2 2^{-n_1-n_2}\}$$

then put

$$\mathcal{L}_2 = \bigcup_{n_1=1}^{\infty} \bigcup_{n_2=1}^{\infty} L(\pm n_1, \pm n_2) .$$

Kth Lace. For $i=1,\ldots,k$, let n_i be positive integers and $e_i = \pm 1$. Define

$$L(e_1 n_1, \ldots, e_k n_k) = \{y = e_1 2^{-n_1} + \ldots + e_k 2^{-2k+2} 2^{-n_1-\ldots-n_k}$$

and put

$$\mathcal{L}_k = \bigcup_{n_1=1}^{\infty} \cdots \bigcup_{n_k=1}^{\infty} L(\pm n_1, \ldots, \pm n_k) .$$

Diamonds. These will be closed sets which form a decreasing sequence.

1st Diamond. Let $\mathfrak{h}$ be the neck of width 1 about the line segment [0,1]. Put

$$\mathcal{D}_1 = \mathfrak{h} .$$

2nd Diamond. The intersection $\mathcal{D}_1 \cap \mathcal{L}_1$ (a necklace!) is a countable collection of line segments, called <u>links</u>, finitely many contained in each $L(e_1 n_1) \in \mathcal{L}_1$. If L is a link on $L(e_1 n_1)$, construct the neck $\mathfrak{h}(L, 2^{-2}2^{-n_1})$; take $\mathcal{D}_2$ to be the union of all such necks, together with $\mathcal{L}_0 \cap \mathcal{D}_1$. Then $\mathcal{D}_2 \subset \mathcal{D}_1$ and $\mathcal{D}_2$ is closed. This is illustrated by Figure 4. (For convenience, the horizontal distances are lengthened in the illustration.)

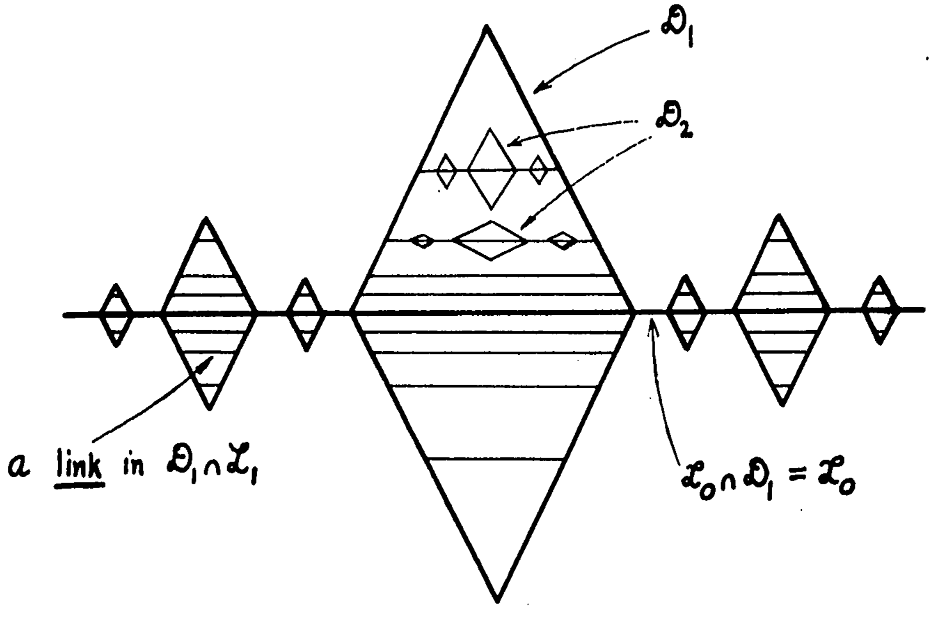


Figure 4.

Kth Diamond. The intersection $\mathcal{D}_{k-1} \cap \mathcal{L}_{k-1}$ is a countable collection of closed line segments, called <u>links</u>, finitely many contained on each $L(e_1n_1,\ldots,e_{k-1}n_{k-1}) \in \mathcal{L}_{k-1}$. If L is a link on $L(e_1n_1,\ldots,e_{k-1}n_{k-1})$, construct the neck

$$\mathfrak{n}(L, 2^{-2k+2}2^{-n_1-\ldots-n_{k-1}})$$

and take as $\mathcal{D}_k$ the union of all such necks plus

$$(\mathcal{D}_1 \cap \mathcal{L}_0) \cup (\mathcal{D}_1 \cap \mathcal{L}_1) \cup \ldots \cup (\mathcal{D}_{k-2} \cap \mathcal{L}_{k-2}) .$$

Then $\mathcal{D}_k \subset \mathcal{D}_{k-1}$ and $\mathcal{D}_k$ is closed.

Significant Properties of the Diamonds.

(5.5) Between any two adjacent lines in $\mathcal{L}_{k-1}$, say

$$L(e_1n_1,\ldots,e_{k-1}n_{k-1}) \quad \text{and} \quad L(e_1n_1,\ldots,e_{k-1}(n_{k-1}+1)) ,$$

there exists an open strip S parallel to $\{y = 0\}$ which meets none of the necks in $\mathcal{D}_k$, i.e., $S \subset \mathcal{D}_k'$. This is illustrated by Figure 5. We say that S is a <u>free strip</u> between the indicated lines.

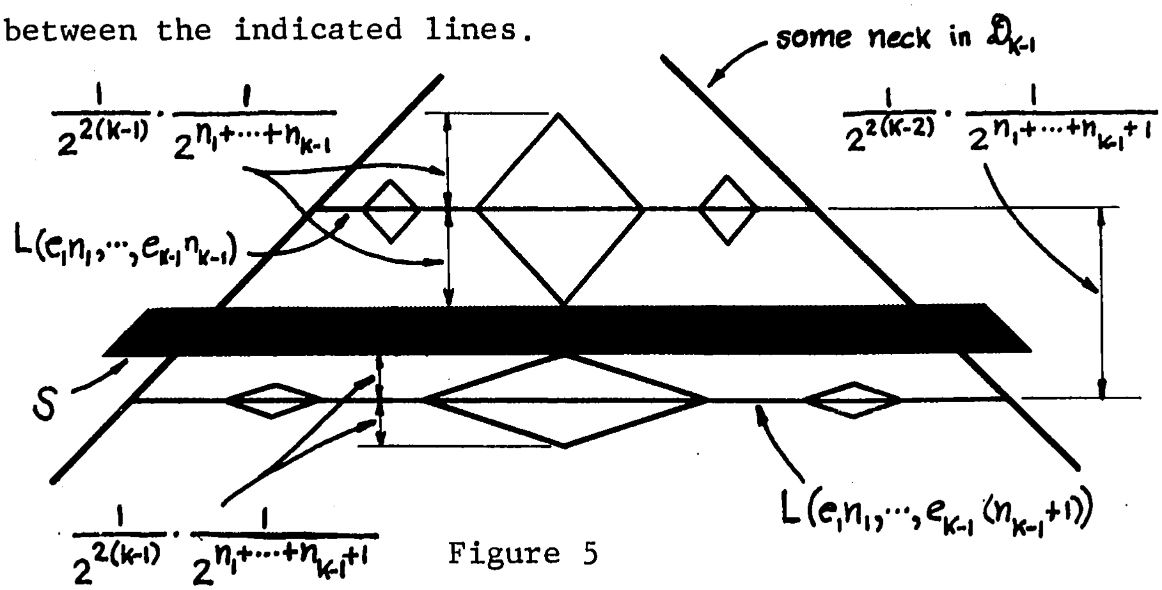


Figure 5

(5.6) $\mathcal{D}_k'$ is a domain. Clearly D_k' is open. Connectedness is proved by induction on k. First, $\mathcal{D}_1'$ is connected. Suppose next that $\mathcal{D}_{k-1}'$ is connected. It suffices to show that any point $P \in \mathcal{D}_k'$ can be joined to a point of $\mathcal{D}_{k-1}'$ by a path $\alpha \subset \mathcal{D}_k'$. Thus, let $P \in \mathcal{D}_{k-1} \cap \mathcal{D}_k'$. Then

$$P \notin (\mathcal{D}_1 \cap \mathcal{L}_o) \cup (\mathcal{D}_1 \cap \mathcal{L}_1) \cup \ldots \cup (\mathcal{D}_{k-2} \cap \mathcal{L}_{k-2}) \cup (\mathcal{D}_{k-1} \cap \mathcal{L}_{k-1}) ,$$

and hence P does not belong to any of the links used in constructing the necks for $\mathcal{D}_{k-1}$. But P must belong to some neck in $\mathcal{D}_{k-1}$, hence must lie strictly between two of the lines in the lace $\mathcal{L}_{k-1}$ used to construct $\mathcal{D}_k$, else $P \in \mathcal{L}_{k-1} \cap \mathcal{D}_{k-1} \subset \mathcal{D}_k$. Let these lines be those of (5.5). By the geometry of the necks, P can be joined to the free strip S by a vertical line in $\mathcal{D}_k'$, and hence, by a path in S, to $\mathcal{D}_{k-1}'$.

The Domain D. Define $\mathcal{D} = \bigcap \mathcal{D}_n$, a set which might as well be called a diamond necklace. Then, put

$$D = T - \mathcal{D} .$$

Since $D = T \cap \bigcup \mathcal{D}_k'$ and the $\mathcal{D}_k'$ are an increasing sequence of domains, D is a domain.

The Boundary of D. Clearly $\Gamma \subset \partial D$; we show that the boundary points of $\partial D - \Gamma$ which are accessible from D form a totally disconnected set. Put $B = \partial D - \Gamma$.

By construction, for any $\mathcal{D}_k$ and any link L used in the construction of $\mathcal{D}_k$, the Cantor set on L is accessible from D, and no other point of L.

Note that if $b \in B$, then there are lines of $\bigcup \mathcal{L}_k$ arbitrarily close to b. From this it follows that any neighborhood of b meets a free strip $S \subset D'$ between two adjacent lines of some $\mathcal{L}_k$. Hence, if $b_1, b_2 \in B$, with $\mathrm{Im}\, b_1 < \mathrm{Im}\, b_2$, then there is a free strip $S \subset D'$ separating b_1 and b_2.

Now $\partial D - \Gamma \subset B$; let $C \subset \partial D - \Gamma$ be a connected set containing more than one point. We will show that C must contain an inaccessible boundary point. By the preceding remark, C must be a subset of some line ℓ parallel to $\{y=0\}$. If $\ell \in \bigcup \mathcal{L}_k$, then $C \subset L$, where L is some link used in the construction of some $\mathcal{D}_k$. But we have seen that only the Cantor set on L is accessible. Thus C must contain inaccessible points.

On the other hand, possibly $\ell \notin \bigcup \mathcal{L}_k$. But then $C \cap \ell$ contains an interval I of positive length, and

$$I \subset B \subset \mathcal{D} = \bigcap \mathcal{D}_k ,$$

so it is possible to determine a nested sequence of necks $\mathfrak{n}_k$, with $I \subset \mathfrak{n}_k \subset \mathcal{D}_k$. But diam $\mathfrak{n}_k \to 0$ as $k \to \infty$, so I cannot have positive length. Hence $C \cap \ell$ must reduce to a point. $\blacksquare$

We conclude this section with a theorem which helps to give a simple characterization of $\mathfrak{C}(f)$ for $f \in \mathcal{U}$ (see (3.13)). The method of proof was used by Ohtsuka [13] to establish another fact about the class $\mathcal{U}$.

THEOREM 5. <u>Let</u> $\mathcal{A}$ <u>be an analytic set with the property</u>

$$\{|w| = 1\} \subset \mathcal{A} \subset \{|w| \le 1\} .$$

<u>There exists</u> $f \in \mathcal{U}$ <u>with</u> $\mathcal{A}(f) = \mathcal{A}$.

Proof. By the Theorem of Lusin and Sierpinski, (3.5), there exists a function $\alpha:(0,1] \to \mathcal{A}$ onto and continuous on the left.

For each finite system of indices $(i_1,\ldots,i_n)$, $i_k = 0,1$, choose a point $P(i_1,\ldots,i_n)$ satisfying

1) $|P(i_1\ldots i_n)| < 1$;
2) $P(0)$, $P(1)$ lie on distinct radii;
3) $P(i_1\ldots i_n)$, $P(i_1\ldots i_n 0)$, $P(i_1\ldots i_n 1)$ lie on distinct radii;
4) $|\alpha(\sum_1^n i_k 2^{-k}) - P(i_1\ldots i_n)| < 2^{-n}$, $n \ge 2$.

For each n, let $R(i_1\ldots i_n)$ be the closed segment of the radius through $P(i_1\ldots i_n)$, of length $\ell(n) < 2^{-n}$, and lying in the annulus $\{|P(i_1\ldots i_n)| \le |z| < 1\}$ (see Figure 6).

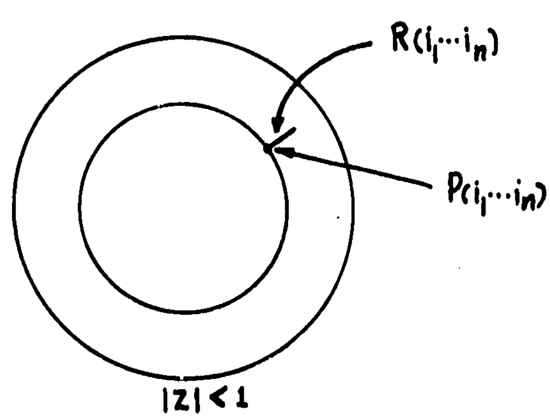


Figure 6.

Let S_o be a copy of the sphere cut along $R(0)$ and $R(1)$. For each finite system $(i_1 \ldots i_n)$ of indices, $i_k = 0,1$, let $S(i_1 \ldots i_n)$ be a copy of the sphere cut along $R(i_1 \ldots i_n)$, $R(i_1 \ldots i_n 0)$, $R(i_1 \ldots i_n 1)$, as in Figure 7.

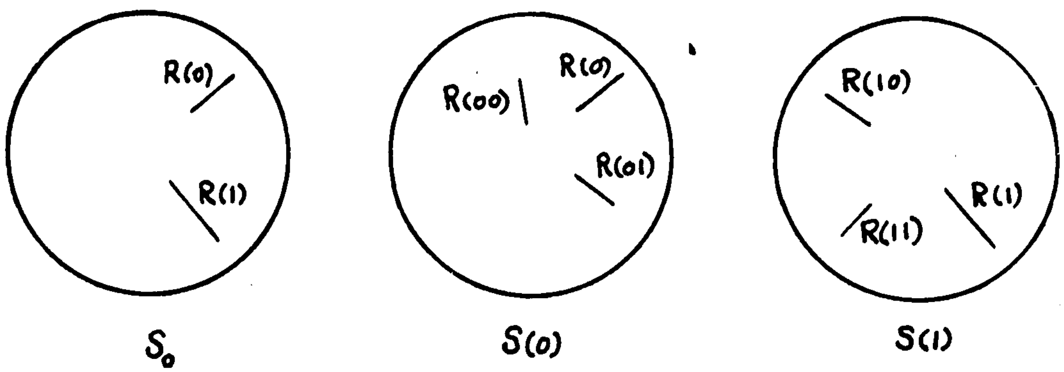


Figure 7.

Join $S(i)$, $i=0,1$, to S_o so as to form first order branch points at the endpoints of $R(0)$, $R(1)$; denote by $\mathcal{S}_1$ the bordered surface so constructed. Now $\mathcal{S}_1$ has four boundary slits, along $R(ij)$, $i,j = 0,1$, so attach to each slit $R(ij)$ of $\mathcal{S}_1$ a copy of $S(ij)$ so that first order branch points are created at the endpoints of $R(ij)$; let $\mathcal{S}_2$ be the bordered surface so obtained. The boundary of $\mathcal{S}_2$ consists of the eight slits $R(ijk)$, $i,j,k = 0,1$. Continuing in this manner we obtain

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3 \subset \dots \quad .$$

Now define

$$\mathcal{S} = \bigcup \mathcal{S}_n \ .$$

It is clear from the construction that $\mathcal{S}$ is a Riemann surface of planar character [1,p.175]. Moreover, by choosing $\ell(n)$ small enough, $\mathcal{S}$ has null boundary [1,pp.204-206]. Hence there exists a 1-1 conformal mapping φ of $\mathcal{S}$ onto a plane domain D in the W-plane whose boundary B is a closed set of harmonic measure zero (since $\mathcal{S}$ has null boundary).

Let $\mho$ be the part of $\mathcal{S}$ over $\{|w| < 1\}$; then $\mho$ is connected and $\varphi(\mho)$ is a subdomain $D_o \subset D$ bounded by B and countably many closed curves C_n, each of which is the image under φ of a simple, closed, schlicht curve Γ_n on $\mathcal{S}$ lying over $\{|w| = 1\}$.

The universal covering surface $\tilde{D}_o$ of D_o is a hyperbolic Riemann surface; let ψ be the 1-1 conformal mapping of $\tilde{D}_o$ onto $\{|z| < 1\}$. Composing $\check{\psi}$ with the projection $p: \tilde{D}_o \to D_o$, we obtain a holomorphic function $h(z)$ from $\{|z| < 1\}$ onto D_o. Similarly composing $\check{\varphi}$ with the projection $\pi: \mho \to \{|w| < 1\}$, we obtain a holomorphic function $g(W)$ from D_o onto $\{|w| < 1\}$. This is illustrated in Figure 8.

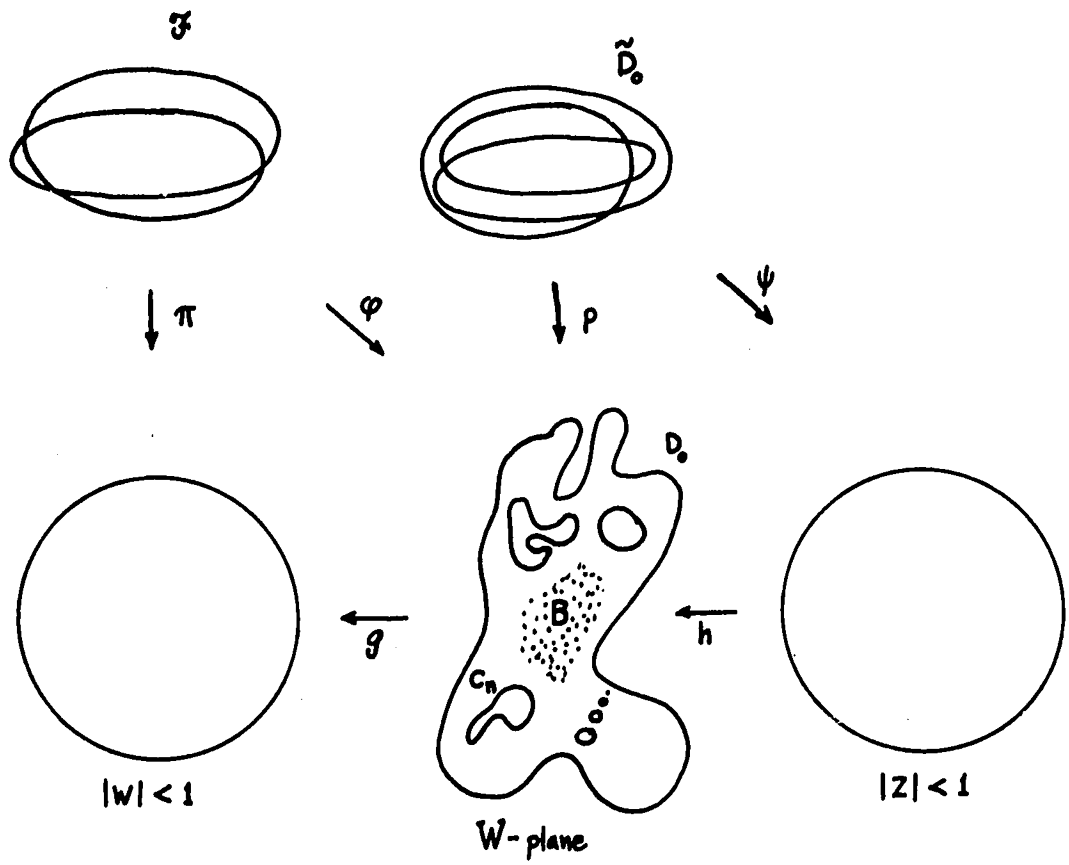


Figure 8.

We show now that the function $f = g \circ h$ has the desired properties. Clearly f maps $\{|z| < 1\}$ onto $\{|w| < 1\}$. By a theorem of Nevanlinna [11,p.209], the set of points on $\{|z| = 1\}$ at which h has radial limits in the set B is of measure zero. Thus h has radial limits lying on the closed curves C_n a.e. on $\{|z| = 1\}$, and it follows that f has radial limits of modulus one a.e. Thus $f \in \mathcal{U}$; it remains to show that $\mathfrak{a}(f) = \mathfrak{a}$.

First, $\mathfrak{a} \subset \mathfrak{a}(f)$: If $a \in \mathfrak{a}$, let $\alpha(x) = a$, where $x = \sum i_k 2^{-k}$, $i_k = 0,1$, is the non-terminating binary expansion of x. Consider the <u>branch</u> points $P(i_1 \ldots i_n)$ on $\mathcal{S}$

determined by the indices $(i_1 i_2 \ldots)$ in this expansion. Construct a polygonal line Γ in $\mathfrak{S}$ as follows: join $P(i_1)$ to $P(i_1 i_2)$ by a line segment in $S(i_1) \cap \mathfrak{S}$; join $P(i_1 i_2)$ to $P(i_1 i_2 i_3)$ by a line segment in $S(i_1 i_2) \cap \mathfrak{S}$, etc. Now the projection $\pi(\Gamma) = \gamma$ lies in $\{|w| < 1\}$ and ends at the point $a \in \{|w| \leq 1\}$: γ is polygonal and passes successively through the points $P(i_1 \ldots i_n)$ with

$$|Pi_1 \ldots i_n) - a| \leq |P(i_1 \ldots i_n) - \alpha(\sum_1^n i_k 2^{-k})|$$

$$+ |\alpha(\sum_1^n i_k 2^{-k}) - a|$$

$$< 2^{-n} + \epsilon_n$$

where $\epsilon_n \to 0$ as $n \to \infty$, since α is continuous on the left. Finally, since Γ is a non-compact arc on $\mathfrak{S}$, Γ corresponds via the map $\psi \circ \check{p}$ to an arc γ in $\{|z| < 1\}$, which tends to a point of $\{|z| = 1\}$, and on which f has the asymptotic value a. Thus $a \in \mathfrak{A}(f)$.

Next, $\mathfrak{A}(f) \subset \mathfrak{A}$: if $a \in \mathfrak{A}(f)$ and $|a| = 1$, then $a \in \mathfrak{A}$ by hypothesis. Thus suppose $|a| < 1$. Then there exists an arc γ in $\{|z| < 1\}$ tending to $\{|z| = 1\}$ mapped by f onto an arc $f(\gamma)$ in $\{|w| < 1\}$ ending at a. By the construction of f, $f(\gamma)$ is the projection of a non-compact arc Γ on $\mathfrak{S}$. Since $|a| < 1$, Γ cannot tend to one of the arcs Γ_n of $\mathfrak{S}$ over $\{|w| = 1\}$. So Γ must meet infinitely many of the branch lines. Since $\pi(\Gamma) = f(\gamma)$ tends to a, Γ meets infinitely many branch lines $R(i_1 \ldots i_n)$ an <u>odd</u> number of times. These $R(i_1 \ldots i_n)$ uniquely determine their endpoints $P(i_1 \ldots i_n)$, which form a sequence of points tending to a. The indices $(i_1 \ldots i_n)$ so determined give rise to a binary expansion

$$x = \sum_1^{\infty} i_k 2^{-k}$$

and $a = \alpha(x)$, since

$$|a-\alpha(x)| \le |a-P(i_1 \ldots i_n)| + |P(i_1 \ldots i_n)-\alpha(\sum_1^n i_k 2^{-k})|$$
$$+ |\alpha(\sum_1^n i_k 2^{-k}) - \alpha(x)| .$$

Thus $a \in \mathcal{C}$, and hence $\mathcal{C}(f) \subset \mathcal{C}$. $\square$

COROLLARY. An analytic set $\mathcal{C}$ is the asymptotic set of a function in the class $\mathcal{U}$ if, and only if, it has the property $\{|w| = 1\} \subset \mathcal{C}^- \subset \{|w| \le 1\}$.

Proof. It is known [12,p.37] that if $f \in \mathcal{U}$, then $\{|w| = 1\} \subset \mathcal{C}(f)^- \subset \{|w| \le 1\}$. Then the theorem completes the proof. $\square$

6. A Characterization. We first characterize the asymptotic sets of holomorphic functions which map $\{|z| < 1\}$ onto $\{|w| < 1\}$. This result is then used to prove Theorem 7, which extends the characterization to arbitrary functions holomorphic in $\{|z| < 1\}$.

THEOREM 6. *Let* $\mathfrak{a}$ *be an analytic set satisfying* $\mathfrak{a} \subset \{|w| \leq 1\}$. *Then* $\mathfrak{a}$ *is the asymptotic set of a function* f *mapping* $\{|z| < 1\}$ *onto* $\{|w| < 1\}$ *if, and only if, for all* r, $0<r<1$, *there exists a holomorphic function* g_r *mapping* $\{|z| < 1\}$ *into* $\{|w| < 1\}$ *with the properties*

(1) g_r *maps a closed Jordan region topologically onto* $\{|w| \leq r\}$,

(2) $\mathfrak{a}(g_r) \subset \mathfrak{a}$.

Proof. Necessity. Let f map $\{|z| < 1\}$ onto $\{|w| < 1\}$ and let $\mathfrak{a}(f)$ be the asymptotic set of f. Then $\mathfrak{a}(f)$ is analytic, (5.1), and $\mathfrak{a}(f)^{-} \subset \{|w| \leq 1\}$. Let r be fixed, $0<r<1$; we obtain g_r by constructing a suitable Riemann surface $\mathcal{G}$.

Preliminaries to the construction. Let $a,b \in \mathfrak{a}(f)$ with $-1 \leq a < -(1-r)$ and $1-r < b \leq 1$. Let α be a simple arc joining a and b with the properties

(1) $\alpha \subset \{y \geq 0\} \cap \{1-r<|w|<1\}$;

(2) α meets $\{y=0\}$ only at a,b ;

(3) α passes through no branch point projections of the Riemann surface $\mathfrak{F}$ associated with the function f , except possibly a and b .

Let β be a simple arc in $\{y \leq 0\} \cap \{1-r<|w|<1\}$ with properties (2) and (3). Then $\alpha \cup \beta$ bounds a Jordan domain J containing $\{|w| \leq r\}$.

Consider first the arc α; $\mathfrak{F}$ covers $\{|w| < 1\}$, so each $w \in \alpha$ is an interior point of a simple open arc $\gamma_w \subset \alpha \cup \beta$, where γ_w is the projection of a simple open arc Γ_w in $\mathfrak{F}$. By the compactness of α, finitely many $\gamma_{w_i} \equiv \gamma_i$, $i=1,\ldots,n$, cover α; we may assume that this finite covering $\{\gamma_i\}$ is non-redundant, that w_i precedes w_{i+1} on α, $i=1,\ldots,n-1$, and that $w_1=a$, $w_n=b$. Finally, we take for Γ_a the piece of Γ_1 lying over α, and for Γ_b the piece of Γ_n lying over α.

Let Λ_1 be the maximal continuation of Γ_a over α. If Λ_1 covers $\alpha[a,b]$, then we are done. Otherwise, Λ_1 determines an asymptotic value $a_1 \in \alpha$, i.e., $a_1 \in \mathfrak{a}(f)$. Since the covering is non-redundant, $a_1 \in \gamma_{i_1}$, $i_1 \geq 2$. Let Λ_2 be the continuation of Γ_{i_1} over $\alpha[a_1,b]$; if Λ_2 covers $\alpha[a_1,b]$, then we are done. Otherwise, Λ_2 determines an asymptotic value $a_2 \in \alpha$, and we have $a_2 \in \gamma_{i_2}$, $i_2 \geq 3$. Now let Λ_3 be the continuation of Γ_{i_2} over $\alpha[a_2,b]$, and repeat the process established above. Since there are only finitely many Γ_i, the process is completed after finitely many steps, say m steps, with Λ_m covering $\alpha[a_m,b]$.

Thus we have obtained a finite sequence of simple arcs in $\mathfrak{F}$, $\Lambda_1,\ldots,\Lambda_m$ whose projections $\lambda_1,\ldots,\lambda_m$ cover α and are disjoint except possibly for endpoints. Moreover, the endpoints of each λ_k are points of $\mathfrak{a}(f)$.

In the same way, we obtain a finite sequence $\Lambda_1,\ldots,\Lambda_p$ of arcs in $\mathfrak{J}$, whose projections $\delta_1,\ldots,\delta_p$ cover β and are disjoint except possibly for endpoints. Moreover, the endpoints of each δ_k are points of $\mathfrak{a}(f)$.

Construction of the Riemann Surface $\mathcal{G}$. J is the Jordan region bounded by $\alpha \cup \beta$ and containing $\{|w| \le r\}$. Let $\mathfrak{J}_1$ be a copy of $\mathfrak{J}$ cut along the arc Λ_1; Λ_1 is not a crosscut of $\mathfrak{J}$ since Λ_1 contains a point over a. Similarly, none of the Λ_n, Δ_n are crosscuts. Now attach the edge Λ_1 of $\mathfrak{J}_1$ to the edge λ_1 of J. Since Λ_1 is not a crosscut, the structure so obtained has a free edge along Λ_1, so additional copies of $\mathfrak{J}_1$ must be attached so as to form logarithmic branch points at the endpoints of λ_1, which are already points of $\mathfrak{a}(f)$. The resulting surface is a bordered Riemann surface, the border resulting from the "free" boundary of J, not from any free edges over λ_1: There are none.

We repeat the construction for each of the edges λ_k, δ_k of J, and so obtain an unbordered Riemann surface $\hat{\mathcal{G}}$. However, $\hat{\mathcal{G}}$ is possibly not simply connected, as would be the case if both endpoints of some Λ_k, Δ_k were points of $\mathfrak{J}$. But the universal covering surface $\mathcal{G}$ of $\hat{\mathcal{G}}$ is a hyperbolic Riemann surface over $\{|w| < 1\}$; we take g_r to be the holomorphic function mapping $\{|z| < 1\}$ onto $\mathcal{G}$ (section 2). From the construction of $\mathcal{G}$,

(1) g_r maps a closed Jordan region topologically onto $\{|w| \leq r\}$,

(2) $\mathfrak{A}(g_r) \subset \mathfrak{A}(f)$.

The last assertion follows from the fact that $\mathfrak{A}(g_r) = \mathfrak{A}(f)$, since in effect $\mathfrak{G}$ "contains" the surface $\mathfrak{S}$.

Sufficiency. Let $\mathfrak{A}$ be an analytic set satisfying $\mathfrak{A}^- \subset \{|w| \leq 1\}$. By hypothesis, for all r, $0<r<1$, there is a holomorphic function g_r mapping $\{|z| < 1\}$ into $\{|w| < 1\}$ with the properties:

(1) g_r maps a closed Jordan region topologically onto $\{|w| \leq r\}$,

(2) $\mathfrak{A}(g_r) \subset \mathfrak{A}$.

We construct first an <u>interior</u> map h (3.14) of a suitable subdomain D of $\{|z| < 2\}$ onto $\{|w| < 1\}$, whose set of asymptotic values is precisely the analytic set $\mathfrak{A}$. Then, using the universal covering surface $\tilde{D}$ of D , we obtain an interior map $\tilde{h}$ of $\{|z| < 1\}$ onto $\{|w| < 1\}$ whose asymptotic set is $\mathfrak{A}$. By the result of (3.14), we can compose $\tilde{h}$ with a homeomorphism of the unit disc to obtain a function f holomorphic from $\{|z| < 1\}$ onto $\{|w| < 1\}$ with $\mathfrak{A}(f) = \mathfrak{A}$. We will use the following two lemmas.

LEMMA 1. <u>Given</u> r, $0<r<1$, <u>there exists a holomorphic function mapping</u> $\{1 < |z| < 2\}$ <u>into</u> $\{|w| < 1\}$, <u>which maps</u> $\{|z| = 1\}$ <u>topologically onto</u> $\{|w| = r\}$, <u>and whose asymptotic values as</u> $|z| \to 2$ <u>are in</u> $\mathfrak{A}$.

To prove Lemma 1, let J be the closed Jordan region in $\{|z| < 1\}$ mapped by g_r topologically onto $\{|w| \leq r\}$. Let $\beta(z)$ be a conformal homeomorphism of

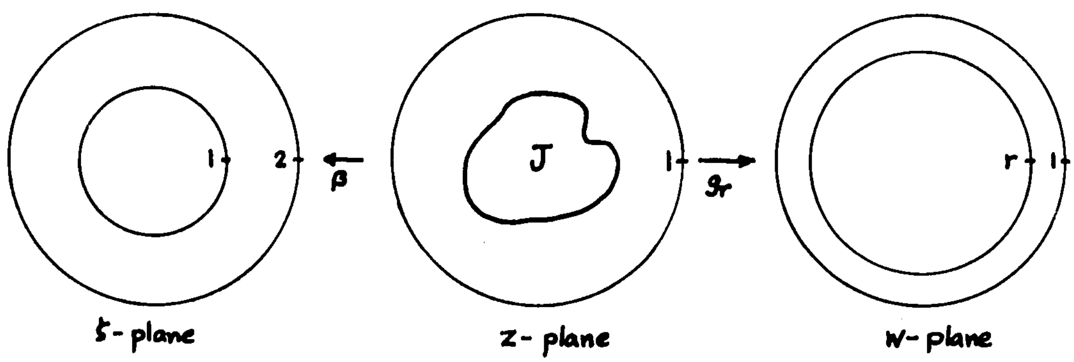


Figure 9.

$\{|z| < 1\} - J$ onto $\{1<|\zeta|<2\}$, which maps ∂J topologically onto $\{|\zeta| = 1\}$. Then $g_r \circ \check{\beta}$ is the desired function (Figure 9).

LEMMA 2. <u>Given</u> r, R_1, R_2, <u>with</u> $0<r<1$, $0<R_1<R_2$, <u>there exists a holomorphic function mapping</u> $\{R_1<|z-z_o|<R_2\}$ <u>into</u> $\{|w| < 1\}$, <u>which maps</u> $\{|z-z_o| = R_2\}$ <u>topologically onto</u> $\{|w| = r\}$, <u>and whose asymptotic values as</u> $|z-z_o| \to R_1$ <u>are in</u> $\mathcal{C}$.

To prove Lemma 2, let $\zeta(t)$ be a conformal homeomorphism of $\{R_1<|t-z_o|<R_2\}$ onto $\{1<|z|<2\}$ which carries $\{|t-z_o| = R_2\}$ onto $\{|z| = 1\}$. Then $g_r \circ \check{\beta} \circ \zeta$ is the desired mapping.

A closed set F . Terminology: The intervals obtained by dividing an arbitrary interval I into five subintervals of equal length will be called the 0th, 1st, 2nd, 3rd, and 4th subintervals of I.

Now define inductively a sequence of closed sets, each consisting of a finite union of disjoint closed intervals, as follows: Take $F_o = [0,1]$; having defined F_{n-1} as a finite union of closed intervals, F_n consists of the 1st and 3rd closed subintervals of each interval in F_{n-1} .

Define

$$F = \bigcap_1^\infty F_n ;$$

then F is closed and

$$F = \{x \in (0,1): \quad x = \sum_1^\infty (2i_k+1)5^{-k}, \quad i_k=0,1\} .$$

Note that each of the 2^n intervals in F_n has for its left endpoint one of the numbers

$$x = \sum_1^n (2i_k+1)5^{-k}, \quad i_k=0,1 .$$

Hence there is a 1-1 correspondence between the intervals in F_n and the set of indices $(i_1 \ldots i_n)$, $i_k=0,1$. This provides a convenient indexing of the intervals used to define F. Thus $I(i_1 \ldots i_n)$ is one of the intervals in F_n .

The Domain D. We will define D as the union of certain "circle" regions in $\{|z|<2\}$. Define $C(i_1 \cdots i_n)$ as the circle with diameter $I(i_1 \cdots i_n)$.

Let $B = \{1<|z|<2\}$.

Let E be the <u>closed</u> circle region bounded by $\{|z|=1\}$ and $C(i_1)$, $i_1 = 0,1$.

With each interval $I(i_1 \cdots i_n)$ we associate three <u>closed</u> circle regions, $Q(i_1 \cdots i_n)$, $S(i_1 \cdots i_n)$, and $T(i_1 \ldots i_n)$

$Q(i_1 \cdots i_n)$ is a <u>closed</u> annulus whose outer boundary is $C(i_1 \cdots i_n)$ and whose inner boundary is a circle $C_1(i_1 \cdots i_n)$ so chosen that $Q(i_1 \cdots i_n)$ is disjoint from $I(i_1 \cdots i_n\, i_{n+1})$, $i_{n+1} = 0,1$.

$S(i_1 \cdots i_n)$ is a <u>closed</u> circle region bounded by $C_1(i_1 \cdots i_n)$, $C(i_1 \cdots i_n 0)$, $C(i_1 \cdots i_n 1)$, and a circle $C_2(i_1 \cdots i_n)$ disjoint from the preceding three circles.

$T(i_1 \cdots i_n)$ is an annulus which contains its outer boundary $C_2(i_1 \cdots i_n)$ but not its inner boundary $C_3(i_1 \cdots i_n)$.

Now take

$$D = B \cup E \cup \bigcup Q(i_1 \cdots i_n) \cup \bigcup S(i_1 \cdots i_n) \cup \bigcup T(i_1 \cdots i_n).$$

It is easy to see that D is a subdomain of $\{|z|<2\}$ with boundary

$$\partial D = \{|z|=2\} \cup F \cup \bigcup C_3(i_1 \cdots i_n).$$

These circle regions are illustrated by Figure 10.

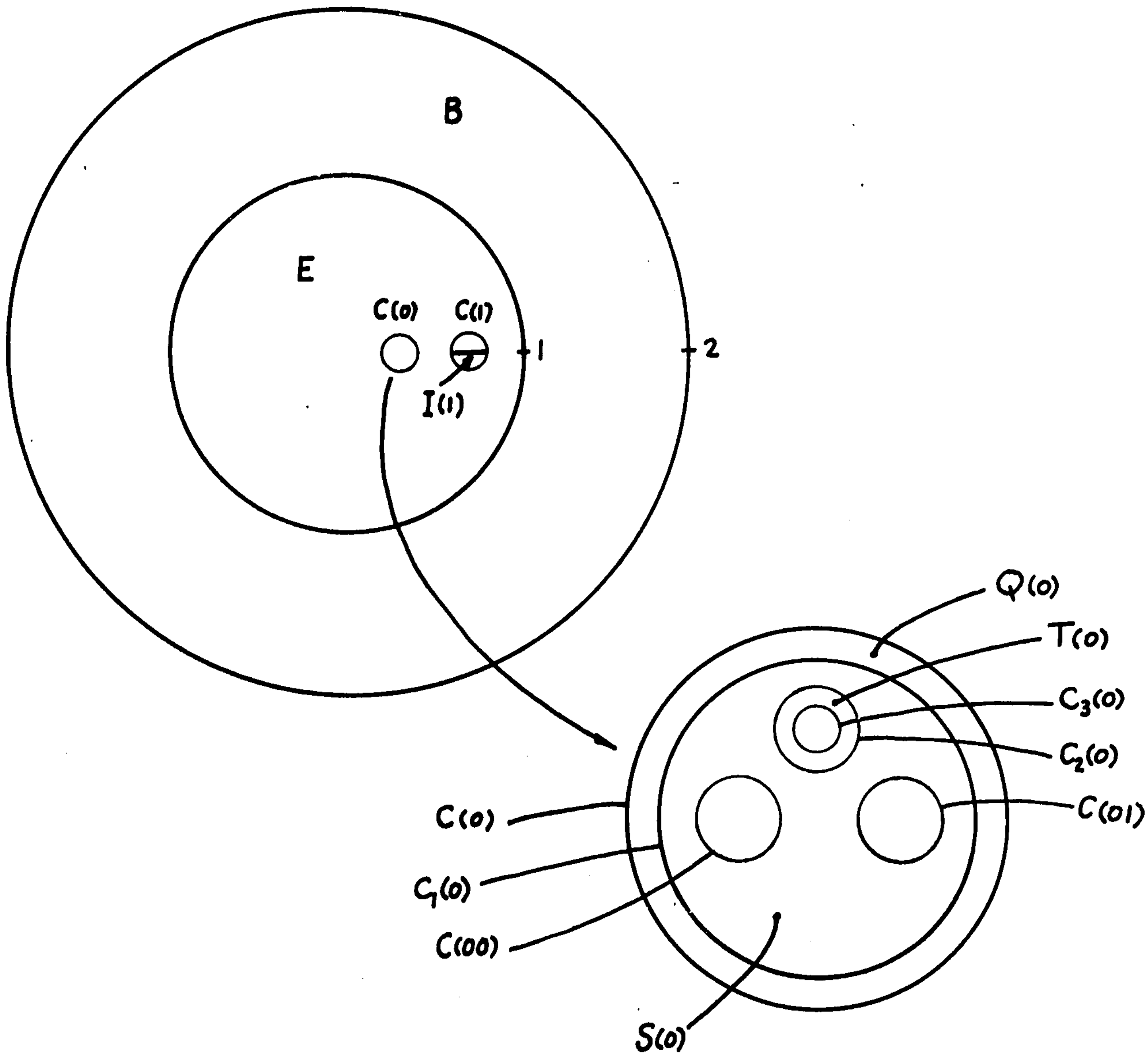


Figure 10.

The interior map h. We will define h by piecing together interior functions defined on the subregions of D discussed above. First, some preliminaries are required.

Let α be a function from $(0,1]$ onto $\mathfrak{a}$ which is continuous on the left (3.5). For each finite set of indices $(i_1 \cdots i_n)$, $i_k = 0,1$, choose a point $P(i_1 \cdots i_n)$ in the complex w-plane satisfying the conditions

(a) $|P(i_1\cdots i_n)| < 1$,

(b) $P(0) \neq P(1)$,

(c) $P(i_1\cdots i_n)$, $P(i_1\cdots i_n 0)$, $P(i_1\cdots i_n 1)$ are distinct,

(d) $|\alpha(\sum_1^n i_k 2^{-k}) - P(i_1\cdots i_n)| < 2^{-n}$, $n \geq 2$.

About each point $P(i_1\cdots i_n)$ construct a circle $C(i_1\cdots i_n)$ of radius $R(i_1\cdots i_n)$ satisfying the conditions

(e) $C^*(i_1\ldots i_n) \subset \{|w|<1\}$, where $C^* = (\text{interior } C) \cup C$,

(f) $C_0^* \cap C_1^* = \phi$

(g) $C^*(i_1\cdots i_n)$, $C^*(i_1\cdots i_n 0)$, $C^*(i_1\cdots i_n 1)$ are disjoint,

(h) $R(i_1\cdots i_n) < 2^{-n}$.

For each $(i_1\cdots i_n)$ let $q(i_1\ldots i_n)$ be an interior mapping of the closed annulus $Q(i_1\cdots i_n)$ onto the disc $C^*(i_1\cdots i_n)$ determined by a homeomorphism from $Q(i_1\cdots i_n)$ to a two-sheeted covering of $C^*(i_1\cdots i_n)$ whose sheets are joined along an interior branch line. $q(i_1\cdots i_n)$ maps $C(i_1\cdots i_n)$ and $C_1(i_1\cdots i_n)$ topologically onto $C(i_1\cdots i_n)$ in such a way that coherent orientations of $C(i_1\cdots i_n)$ and $C_1(i_1\cdots i_n)$ with respect to $Q(i_1\cdots i_n)$ are transformed by $q(i_1\cdots i_n)$ into the <u>same</u> orientation of $C(i_1\cdots i_n)$.

Choose r_1, $0<r_1<1$, so that $\{|w|<r_1\}$ contains $C^*(0)$, $C^*(1)$. By Lemma 1, there exists a holomorphic function $b(z)$ of B into $\{|w|<1\}$ which maps $\{|z|=1\}$ topologically onto $\{|w|=r_1\}$ and whose asymptotic values as $|z| \to 2$ are in $\mathcal{C}$.

Define $h \equiv b$ on B.

Next, there exists a homeomorphism e of E onto

$$(\{|w|\leq r_1\} - C^*(0) \cup C^*(1))^-$$

with the properties

$$e = b \quad \text{on} \quad \{|z|=1\},$$
$$= q(0) \quad \text{on} \quad C(0),$$
$$= q(1) \quad \text{on} \quad C(1).$$

This well known extension of homeomorphisms can be obtained in [16]. This is illustrated by Figure 11.

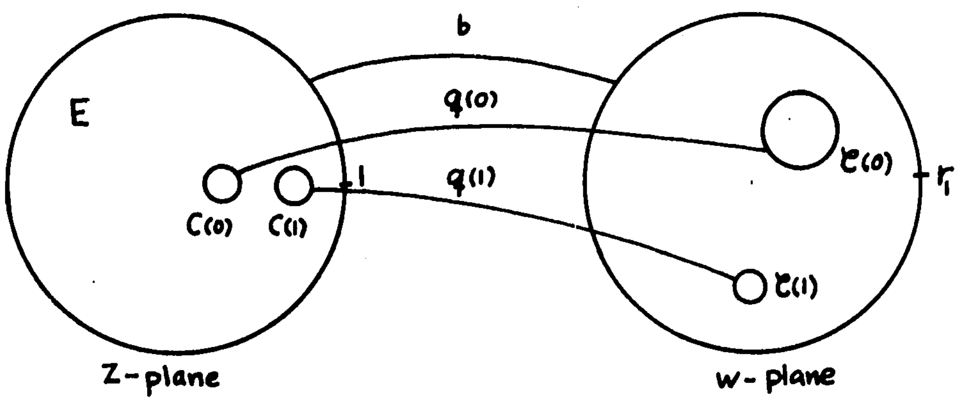


Figure 11.

Define $h \equiv e$ on E.

Define $h \equiv q(i_1 \cdots i_n)$ on $Q(i_1 \cdots i_n)$.

To define h on $S(i_1 \cdots i_n)$ and $T(i_1 \cdots i_n)$, choose r_n, $0<r_n<1$, so that $\{|w|<r_n\}$ contains $\mathfrak{C}^*(i_1 \cdots i_n)$, $\mathfrak{C}^*(i_1 \cdots i_n 0)$, and $\mathfrak{C}^*(i_1 \cdots i_n 1)$. By Lemma 2 there exists a holomorphic function $t(i_1 \cdots i_n)$ mapping $T(i_1 \cdots i_n)$ into $\{|w|<1\}$ which maps $C_2(i_1 \cdots i_n)$ topologically onto $\{|w|=r_n\}$ and whose asymptotic values as $z \to C_3(i_1 \cdots i_n)$ are in $\mathfrak{C}$.

Define $h \equiv t(i_1 \cdots i_n)$ on $T(i_1 \cdots i_n)$.

Now we have the following topological mappings defined on the boundary of $S(i_1 \cdots i_n)$ (Figure 12):

$q(i_1 \cdots i_n)$ from $C_1(i_1 \cdots i_n)$ to $\mathfrak{C}(i_1 \cdots i_n)$

$q(i_1 \cdots i_n 0)$ from $C(i_1 \cdots i_n 0)$ to $\mathfrak{C}(i_1 \cdots i_n 0)$

$q(i_1 \cdots i_n 1)$ from $C(i_1 \cdots i_n 1)$ to $\mathfrak{C}(i_1 \cdots i_n 1)$

$t(i_1 \cdots i_n)$ from $C_2(i_1 \cdots i_n)$ to $\{|w|=r_n\}$,

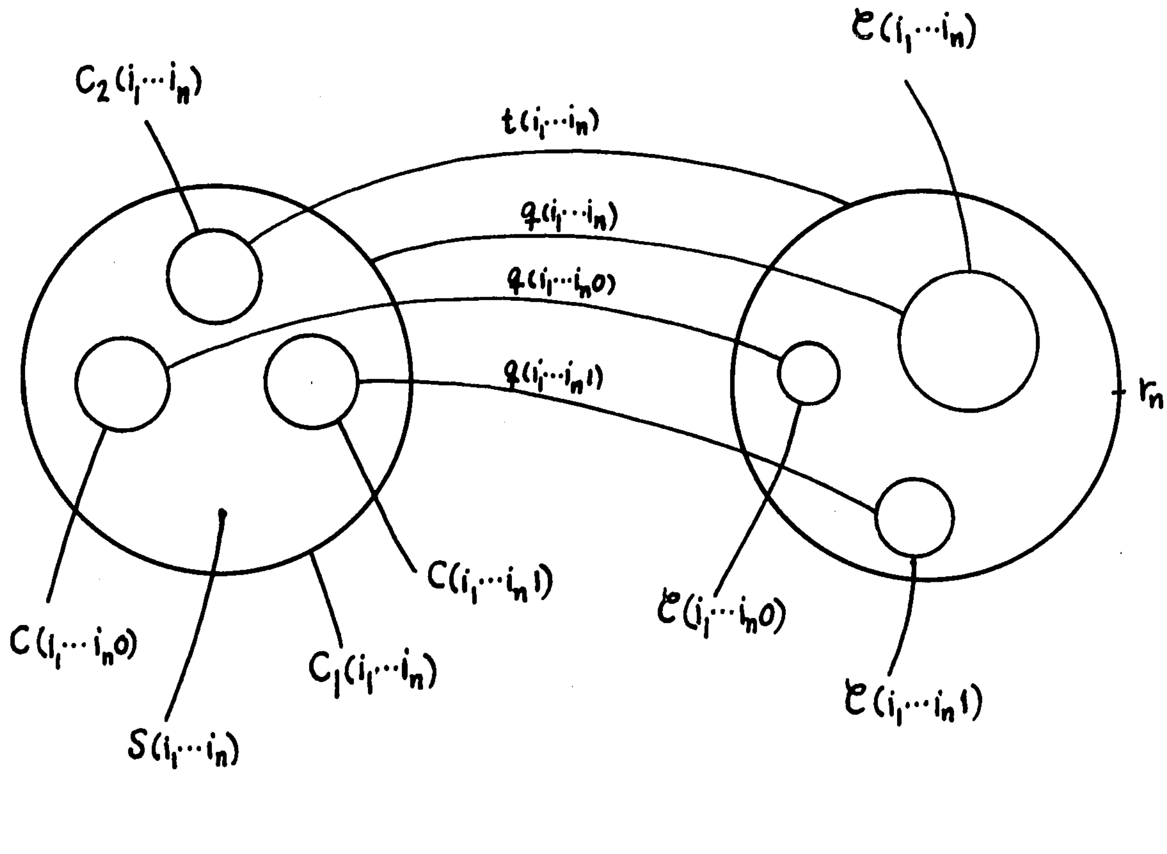


Figure 12.

These homeomorphisms can be extended [16] to a homeomorphism $s(i_1 \cdots i_n)$ of $S(i_1 \cdots i_n)$ onto

$(\{|w| \le r_n\} - \mathbf{C}^*(i_1\cdots i_n) \cup \mathbf{C}^*(i_1\cdots i_n 0) \cup \mathbf{C}^*(i_1\cdots i_n 1))^-$.

Define $h \equiv s(i_1\cdots i_n)$ on $S(i_1\cdots i_n)$.

Thus h is defined on all of D. To verify that h is an interior map, note that h clearly has this property on the interiors of the distinguished subregions of D. On the common boundaries of these subregions h is defined by a homeomorphism which is the restriction of the interior mappings of the abutting subregions. It remains to verify that these homeomorphisms defined on the circles $\{|z| = 1\}$, $C(i_1\cdots i_n)$, $C_1(i_1\cdots i_n)$, $C_2(i_1\cdots i_n)$ induce compatible orientations on the circles $\{|w| = r_n\}$, $\mathbf{C}(i_1\cdots i_n)$. But this is straightforward: An orientation of $\{|z| = 1\}$ induces an orientation on each $C(i_1\cdots i_n)$, $C_1(i_1\cdots i_n)$, $C_2(i_1\cdots i_n)$ as well as induces an orientation of $\{|w| = r_1\}$ through the mapping b. Now this orientation of $\{|w| = r_1\}$ induces orientations on all circles $\{|w| = r_n\}$, $\mathbf{C}(i_1\cdots i_n)$ which are compatible with the orientations induced there by the mappings e, $q(i_1\cdots i_n)$, $s(i_1\cdots i_n)$ and $t(i_1\cdots i_n)$.

Finally, the hypothesis implies that $\{|w| = 1\} \subset \mathcal{A}^-$, so that there exist $\mathbf{C}(i_1\cdots i_n)$ arbitrarily close to $\{|w| = 1\}$. Hence in defining the maps $t(i_1\cdots i_n)$ we use r_n arbitrarily close to 1, so that h maps D <u>onto</u> $\{|w| < 1\}$.

The asymptotic values of h. First, $\mathcal{A}(h) \subset \mathcal{A}$. If $a \in \mathcal{A}(h)$, then there exists a non-compact arc γ in D which is mapped by h into an arc $h(\gamma)$ tending to a.

γ ends at a <u>point</u> $\zeta \in \partial D$: Otherwise γ^- contains <u>distinct</u> points of ∂D, and, by the way h was constructed, this contradicts the fact that $h(\gamma)$ tends to a point. Now if $\zeta \in \{|z|=2\}$, or if ζ belongs to one of the circles $C_3(i_1 \cdots i_n)$, then Lemma 1 or Lemma 2 assures us that $a \in \mathcal{C}$. The remainder of ∂D is F; if $\zeta \in F$, then

$$\zeta = \sum_1^\infty (2i_k+1)5^{-k}, \quad i_k = 0,1,$$

and the finite sets of indices $(i_1 \cdots i_n)$ determined by this expansion correspond to annulii $Q(i_1 \cdots i_n)$ which tend to ζ. Since γ must meet all these $Q(i_1 \cdots i_n)$, $h(\gamma)$ must have a point $w_n \in C^*(i_1 \cdots i_n)$, with $w_n \to a$ as $n \to \infty$. But then

$$a = \alpha\left(\sum_1^\infty i_k 2^{-k} \right),$$

since

$$\begin{aligned} \left|a - \alpha\left(\sum_1^\infty i_k 2^{-k}\right)\right| &\le |a-w_n| + (w_n - P(i_1 \cdots i_n)| \\ &+ \left|P(i_1 \cdots i_n) - \alpha\left(\sum_1^n i_k 2^{-k}\right)\right| \\ &+ \left|\alpha\left(\sum_1^n i_k 2^{-k}\right) - \alpha\left(\sum_1^\infty i_k 2^{-k}\right)\right|. \end{aligned}$$

Each of the terms on the right hand side tend to zero as $n \to \infty$. Hence $a \in \mathcal{C}$.

Next, $\mathcal{C} \subset \mathcal{C}(h)$. If $a \in \mathcal{C}$, then $a = \alpha(x)$, where

$$x = \sum_1^\infty i_k 2^{-k}, \quad i_k = 0,1;$$

thus x corresponds to a unique value $y \in F$ given by

$$y = \sum_{1}^{\infty} (2i_k+1)5^{-k}.$$

Using the $(i_1 \cdots i_n)$ determined by the expansion of x, we construct a polygonal line L in $\{|w|<1\}$ joining in succession $P(i_1)$, $P(i_1 i_2)$, $\cdots$. Since

$$|a - P(i_1 \cdots i_n)| \le |a - \alpha(\sum_{1}^{n} i_k 2^{-k})| + |\alpha(\sum_{1}^{n} i_k 2^{-k}) - P(i_1 \cdots i_n)|,$$

it follows that L must end at a. But one of the pre-images ℓ of L by h is an arc in D tending to y. To see this, simply piece ℓ together from the crosscuts of the various $Q(i_1 \cdots i_n)$ and $S(i_1 \cdots i_n)$ which are mapped by h into L. Here of course we consider only those Q's and S's which are determined by the expansion of x, since they collapse on y. Thus h has the asymptotic value a on ℓ, so $a \in \mathfrak{a}(h)$.

We have obtained $\mathfrak{a} = \mathfrak{a}(h)$. As remarked at the outset of the proof, f is obtained by considering the universal covering surface of D and a suitable homeomorphism. In view of (3.6), we have $\mathfrak{a}(f) = \mathfrak{a}(h)$, and the proof is complete. ▯

Before stating and proving the main theorem, we need a final

LEMMA. *Let* $\mathfrak{a} \subset \{|w|<r\}$, $0<r<\infty$, *be an analytic set. If* g *is holomorphic in* $\{|z|<1\}$ *and maps a Jordan region topologically onto* $\{|w| \le r\}$, *then there exists a function* f *holomorphic in* $\{|z|<1\}$, *with* $\mathfrak{a}(f) = \mathfrak{a}(g) \cup \mathfrak{a}$, *whose*

Riemann surface $\mathfrak{F}$ contains a schlicht sheet over $\{|w|<\frac{1}{2}r\}$.

Proof. We obtain f by the construction of its Riemann surface $\mathfrak{F}$. First, some preliminaries. As usual we appeal to the Theroem of Lusin and Sierpinski (3.5). Thus, let $\alpha : (0,1] \to \mathfrak{C}$ be onto and continuous on the left. For each $x \in (0,1]$, we have a unique non-terminating binary expansion:

$$x = \sum_1^\infty i_n 2^{-n}, \quad i_n = 0,1.$$

For each finite set of indices $(i_1 \cdots i_n)$, $i_k = 0,1$ we define a point $P(i_1 \cdots i_n)$ with the properties

(1) $|P(i_1 \cdots i_n)| < r$

(2) $\operatorname{Arg} P_0 \neq \operatorname{Arg} P_1$

(3) $\operatorname{Arg} P(i_1 \cdots i_n)$, $\operatorname{Arg} P(i_1 \cdots i_n 0)$, $\operatorname{Arg} P(i_1 \cdots i_n 1)$ are pair-wise distinct.

(4) $|P(i_1 \cdots i_n) - \alpha(\sum_{k=1}^n i_k 2^{-k})| < 2^{-n}$, $n \geq 2$.

With each $P(i_1 \cdots i_n)$ we associate a closed segment $S(i_1 \cdots i_n)$ on the radius through $P(i_1 \cdots i_n)$, of length $< 2^{-n}$, with $P(i_1 \cdots i_n)$ one of its endpoints, and contained in $\{|w|<r\}$.

The Riemann surface $\mathfrak{G}$ of g contains a schlicht domain D over $\{|w|\leq r\}$. For each finite set of indices $(i_1 \cdots i_n)$, let $\mathfrak{G}(i_1 \cdots i_n)$ be a copy of $\mathfrak{G}$ whose schlicht domain $D(i_1 \cdots i_n)$ has been severed along the three disjoint seqments $S(i_1 \cdots i_n)$, $S(i_1 \cdots i_n 0)$, $S(i_1 \cdots i_n 1)$. Let $\mathfrak{G}_0$ be a copy of $\mathfrak{G}$ whose schlicht domain D_0 has been cut along the segments $S(0)$, $S(1)$, and a radial segment R, disjoint from $S(0)$, $S(1)$, lying in the annulus $\{\frac{r}{2} < |z| < r\}$.

Let $\mathcal{G}_{-1}$ be a copy of $\mathcal{G}$ whose schlicht domain D_{-1} has been cut along the segment R.

Construction of $\mathfrak{F}$. This is simply a matter of joining the slit copies of $\mathcal{G}$ together in a natural way, then taking the universal covering surface of the Riemann surface so obtained.

0th level. Let $\hat{\mathfrak{F}}_0$ be the structure obtained by joining $\mathcal{G}_{-1}$ and $\mathcal{G}_0$ together so as to form first order branch points at the endpoints of R, which then forms a branch line. $\hat{\mathfrak{F}}_0$ has a schlicht sheet over $\{|w| \le \frac{r}{2}\}$, and free edges along the cuts $S(0)$, $S(1)$ in $\mathcal{G}_0$.

1st level. To the free edges of $\hat{\mathfrak{F}}_0$ along $S(i_1)$ attach a single copy of $\mathcal{G}(i_1)$ so that first order branch points are created at the endpoints of $S(i_1)$. Denote the resulting structure $\hat{\mathfrak{F}}_1$; it has free edges along the four cuts $S(i_1 i_2)$, $i_k = 0,1$.

Nth level. To the free edges of $\hat{\mathfrak{F}}_{N-1}$ along $S(i_1 \cdots i_N)$ attach a copy of $\mathcal{G}(i_1 \cdots i_N)$ so that first order branch points are formed at the endpoints of $S(i_1 \cdots i_N)$. Denote the resulting bordered surface $\hat{\mathfrak{F}}_N$; it has 2^{N+1} cuts $S(i_1 \cdots i_{N+1})$, each with two free edges.

Now take $\hat{\mathfrak{F}} = \bigcup \hat{\mathfrak{F}}_N$. $\hat{\mathfrak{F}}$ has a schlicht disc over $\{|w| \le \frac{1}{2}r\}$, since $\hat{\mathfrak{F}}_0$ has one. Let $\mathfrak{A}(\hat{\mathfrak{F}})$ be the set of asymptotic values determined by asymptotic paths on $\hat{\mathfrak{F}}$ (see(3.6)). We want to show that

$$\mathfrak{A}(\hat{\mathfrak{F}}) = \mathfrak{A}(g) \cup \mathfrak{A}.$$

First, $\mathfrak{A}(\hat{\mathfrak{F}}) \subset \mathfrak{A}(g) \cup \mathfrak{A}$: Let $a \in \mathfrak{A}(\hat{\mathfrak{F}})$; then there is a non-compact arc Λ on $\hat{\mathfrak{F}}$ whose projection $\pi(\Lambda)$ into the w-plane tends to a. If Λ is contained in only finitely many $\hat{\mathfrak{F}}_N$, then Λ must end in some attached copy $\mathfrak{G}(i_1 \ldots i_n)$ of $\mathfrak{G}$ and thus determine a as an asymptotic value in $\mathfrak{A}(g)$. On the other hand, suppose Λ meets infinitely many of the branch lines $S(i_1 \ldots i_n)$ of $\hat{\mathfrak{F}}$. Let $Q(i_1 \ldots i_N)$ be the last intersection of Λ with $\hat{\mathfrak{F}}_{N-1}$ (hence $Q(i_1 \ldots i_N) \in$ some $S(i_1 \ldots i_N)$), and observe that the lengths of the $S(i_1 \ldots i_n)$ become small as $n \to \infty$. Now let

$$x = \sum_1^\infty i_n 2^{-n}$$

be the point of $(0,1]$ determined by the indices of the successive $Q(i_1 \ldots i_N)$. Then $a = \alpha(x)$, i.e., $a \in \mathfrak{A}$, since

$$\begin{aligned} |a - \alpha(x)| \le |a - Q(i_1 \ldots i_n)| &+ |Q(i_1 \ldots i_n) - P(i_1 \ldots i_n)| \\ &+ |P(i_1 \ldots i_n) - \alpha(\sum_1^n i_k 2^{-k})| \\ &+ |\alpha(\sum_1^n i_k 2^{-k}) - \alpha(\sum_1^\infty i_k 2^{-k})| \, . \end{aligned}$$

Thus, $\mathfrak{A}(\hat{\mathfrak{F}}) \subset \mathfrak{A}(g) \cup \mathfrak{A}$.

For the reverse inclusion, it is clear from the construction of $\hat{\mathfrak{F}}$ that $\mathfrak{A}(g) \subset \mathfrak{A}(\hat{\mathfrak{F}})$. If $a \in \mathfrak{A}$, then $a = \alpha(x)$ for some

$$x = \sum_1^\infty i_k 2^{-k} \, , \quad i_k = 0,1 \; ;$$

then the polygonal line L, obtained by joining the successive branch points $P(i_1)$, $P(i_1 i_2) \cdots$, determined by the expansion of x, provides us with a non-compact arc on $\hat{\mathfrak{F}}$ determining a as an asymptotic value. Hence, $\mathfrak{A} \subset \mathfrak{A}(\hat{\mathfrak{F}})$.

To complete the proof, let $\mathfrak{F}$ be the universal covering surface of $\hat{\mathfrak{F}}$. $\mathfrak{F}$ contains a schlicht sheet over $\{|w| \le \tfrac{1}{2}r\}$, since $\hat{\mathfrak{F}}$ does. In view of (3.6), the associated holomorphic function in $\{|z| < 1\}$ has the set $\mathfrak{A}(g) \cup \mathfrak{A}$ as its asymptotic set $\mathfrak{A}(f)$. $\blacksquare$

We now state and prove the main theorem.

THEOREM 7. *Let* $\mathfrak{A}$ *be a subset of the extended complex plane.* $\mathfrak{A}$ *is the set of asymptotic values of a function* f *holomorphic in* $\{|z| < 1\}$ *if, and only if,* $\mathfrak{A}$ *is an analytic set and either* $\infty \in \mathfrak{A}$ *or*

(1) *there exists a simply connected domain* D_o *with the properties*

(a) $\infty \notin D_o$,

(b) $\partial D_o \subset \mathfrak{A}^- \subset D_0^-$,

(c) *(inaccessible* ∂D_o*)* $\cap\, \mathfrak{A} = \emptyset$;

(2) *given any compact connected subset* K *of* D_o, *there exists a holomorphic function* g_k *mapping* $\{|z| < 1\}$ *into* D_o *with the properties*

(d) g_k *maps a Jordan region topologically onto a Jordan region containing* K,

(e) $\mathfrak{A}(g_k) \subset \mathfrak{A}$.

Proof. Necessity. Let f be holomorphic in $\{|z| < 1\}$. The asymptotic set $\mathcal{A}(f)$ of f is analytic (5.1) in the extended plane. Suppose $\infty \notin \mathcal{A}(f)$. By Theorem 4, $\mathcal{A}(f)$ necessarily satisfies (1). Now let K be a compact, connected subset of D_o. Then (2) will follow just as in the necessity proof of Theorem 6 if we can show there exists a Jordan arc $\mathcal{B}$ in $f(\{|z| < 1\})$ which bounds a Jordan region in D_o containing K.

We put $D = f(\{|z| < 1\})$ and utilize in general the notation established in Theorem 4 to prove the existence of D_o.

Observe first that the component C of ∂D containing ∂D_o is precisely ∂D_o. If this were not the case, then $C - \partial D_o$ would not be void. Now $D^- \subset D_o^-$ implies $(C-\partial D_o) \subset D_o$, so that $(C - \partial D_o) \cap D_n^* \neq \phi$ for some n. But this implies $C \cap D_n \neq \phi$, and since $D_n \subset D$, we have a contradiction. Thus $C = \partial D_o$.

Now let h be the parallel slit mapping of D onto the plane less certain <u>bounded</u> horizontal slits. h is a conformal homeomorphism and carries ∂D_o into a slit S_o. There are two cases to consider: either $K \cap D = \phi$ or $K \cap D \neq \phi$.

If $K \cap D = \phi$, the connectedness of K implies the existence of a unique component D_1' of D' containing K. The boundary C_1 of D_1' corresponds to a slit S_1. We have $C_1 \neq \partial D_o$, since otherwise $K \subset D_o'$, hence $S_1 \cap S_o = \phi$.

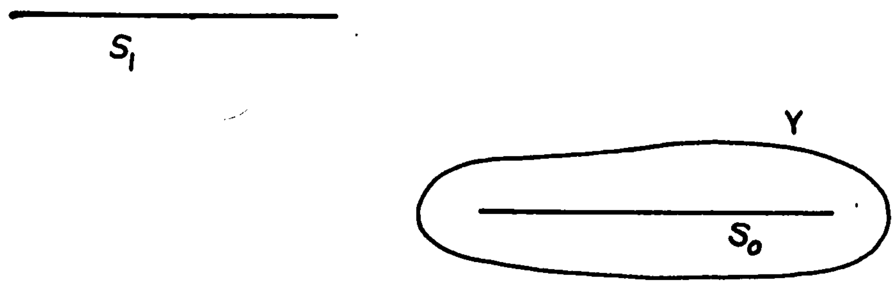


Figure 13.

We may then construct a closed Jordan arc γ in $h(D)$ which separates S_o and S_1 (Figure 13). The Jordan arc $\beta = \check{h}(\gamma)$ in D separates C_1 from ∂D_o, hence bounds a Jordan region in D_o containing K.

If $K \cap D \neq \phi$, we must have $h(K \cap D)$ bounded away from S_o. Otherwise, there exist $a_n \in K \cap D$ such that $\lim h(a_n) \in S_o$. We may assume $\lim a_n = a_o$ exists; put $b_n = h(a_n)$ and $\lim b_n = b_o$. We must have $a_o \in K \cap \partial D$. Let C_2 be the component of ∂D containing a_o, and let C_2 correspond to a slit S_2 under the mapping h. Since $\mathrm{dist}(K,\partial D_o) > 0$, $C_2 \neq \partial D_o$, so that $S_2 \cap S_o = \phi$. Let γ_2 be a closed Jordan arc in $h(D)$ which separates S_2 and S_o, and let γ_o be a closed Jordan arc in $h(D)$ which separates γ_2 and S_o. This is illustrated by Figure 14.

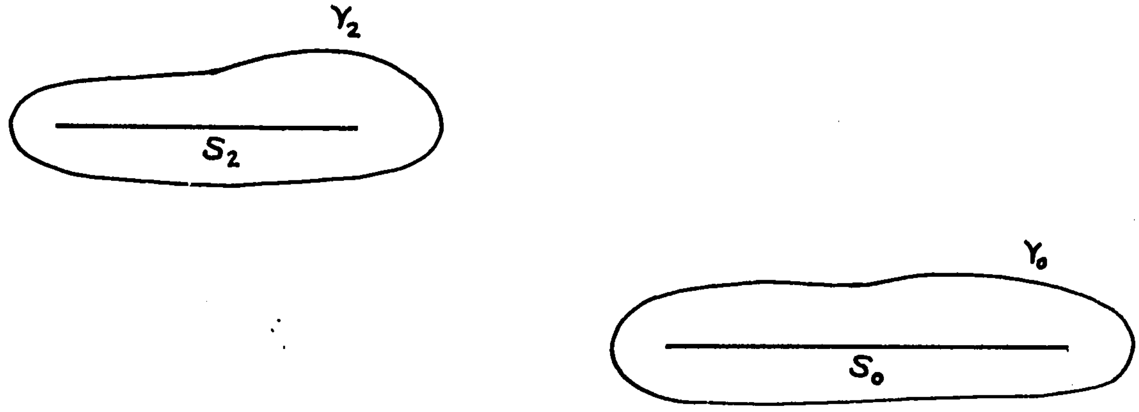


Figure 14.

Now $\check{h}$ carries γ_o, γ_2 into Jordan arcs β_o, β_2 in D with the properties that β_2 separates C_2 and ∂D_o, while β_o separates β_2 and ∂D_o. Then the subdomain $\mathcal{D}_2$ of D bounded by C_2, β_2 must be disjoint from the one, $\mathcal{D}_o$, bounded by β_o, ∂D_o. Then, $h(\mathcal{D}_o) \cap h(\mathcal{D}_2) = \phi$. But $\mathcal{D}_2$ contains all a_n, except possibly finitely many, since $\lim a_n = a_o \in C_2$, so that $h(\mathcal{D}_2)$ contains all but finitely many b_n. But since $\lim b_n = b_o \in S_o$, $h(\mathcal{D}_o)$ must contain all but finitely many b_n. Thus $h(\mathcal{D}_o) \cap h(\mathcal{D}_2) \neq \phi$, a contradiction.

Thus, $h(K \cap D)$ is bounded away from S_o. It is then possible to construct a closed Jordan arc γ in $h(D)$ which separates $h(K \cap D)$ and S_o. Then $\beta = \check{h}(\gamma)$ is the desired arc in D which bounds a Jordan region in D_o containing K.

The rest of the necessity proof mimics that of Theorem 6 and will not be repeated here.

Sufficiency. In view of Theorem 3, it suffices to consider an analytic set $\mathcal{A}$, not containing ∞, for which conditions (1) and (2) hold. We seek a function f holomorphic in $\{|z| < 1\}$ whose asymptotic set is precisely $\mathcal{A}$. There are two cases to consider.

1. D_o is hyperbolic. Let h be a conformal homeomorphism of D_o onto $\{|w| < 1\}$. Let γ_n be the closed Jordan arc in D_o which h carries onto $\{|w| = 1-\frac{1}{n}\}$, $n = 2,3,\cdots,$ then $\gamma_n^* = \gamma_n \cup$ (subdomain of D_o bounded by γ_n) is compact. By condition (2), there exists a function g_n holomorphic in $\{|z| < 1\}$ which maps a Jordan region topologically onto γ_n^* and has $\mathcal{A}(g_n) \subset \mathcal{A}$. For each n, define $f_n = h \circ g_n$ (Figure 15); then f_n is a holomorphic function mapping $\{|z| < 1\}$ into $\{|w| < 1\}$ and carrying a

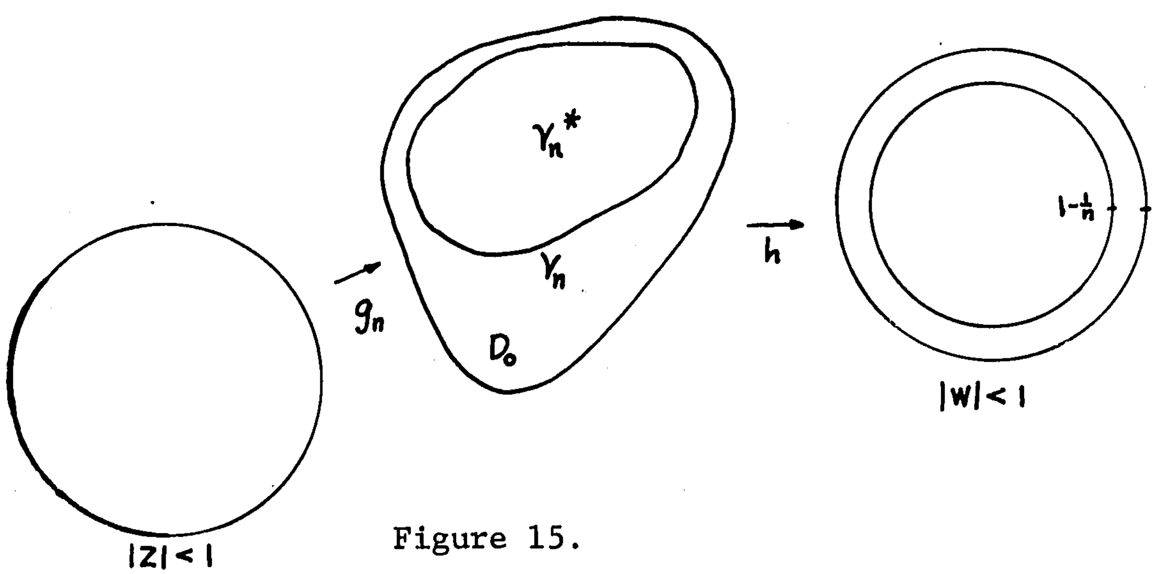


Figure 15.

Jordan region topologically onto $\{|w| \le 1-\frac{1}{n}\}$. Since $\mathcal{A} \cap \partial D_o$ is an analytic set consisting only of accessible boundary points, the boundary correspondence associated with h casts $\mathfrak{a} \cap \partial D_o$ into an <u>analytic</u> subset B of $\{|w| = 1\}$, by Corollary 1 of Theorem 1. Moreover, h carries the analytic set $\mathfrak{a} \cap D_o$ into an <u>analytic</u> set $E \subset \{|w| < 1\}$. Define

$$S = \bigcup_1^\infty \mathfrak{a}(f_n) \cup E \cup B;$$

then S is an analytic subset of $\{|w| < 1\}$ with $S^- \subset \{|w| \le 1\}$. By Theorem 6, there exists a holomorphic function F mapping $\{|z| < 1\}$ <u>onto</u> $\{|w| < 1\}$ with $\mathfrak{a}(F) = S$. Now define $f = \check{h} \circ F$; then f maps $\{|z| < 1\}$ <u>onto</u> D_o, and it will be shown that $\mathfrak{a}(f) = \mathfrak{a}$.

First, $\mathfrak{a}(f) \subset \mathfrak{a}$. Let $a \in \mathfrak{a}(f)$. Since D_o is hyperbolic, f is a normal function, so that there is an arc $\lambda \subset \{|z| < 1\}$ ending at a <u>point</u> of $\{|z| = 1\}$ which is carried by f into an arc Λ in D_o tending to a.

If $a \in D_o$, then $F = h \circ f$ must have the asymptotic value $h(a)$ on λ. That is,

$$h(a) \in \bigcup \mathfrak{a}(f_n) \cup E \cup B;$$

the only interesting possibility is $h(a) \in \mathfrak{a}(f_n)$. Here there exists an arc $\lambda_1 \subset \{|z| < 1\}$ ending at a <u>point</u> of

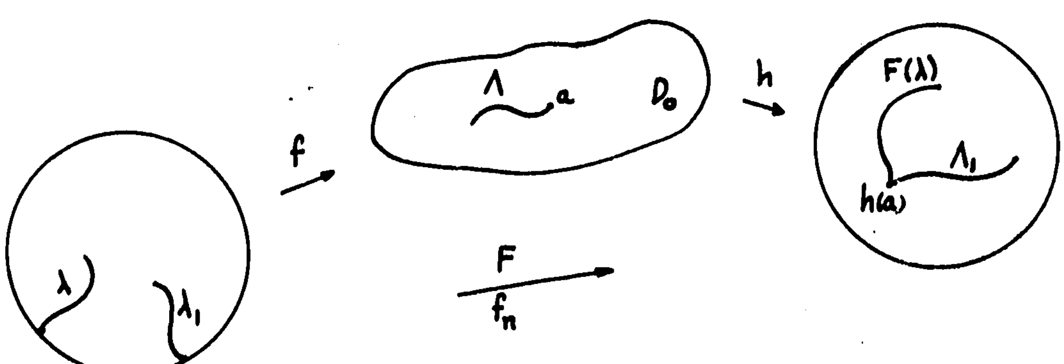
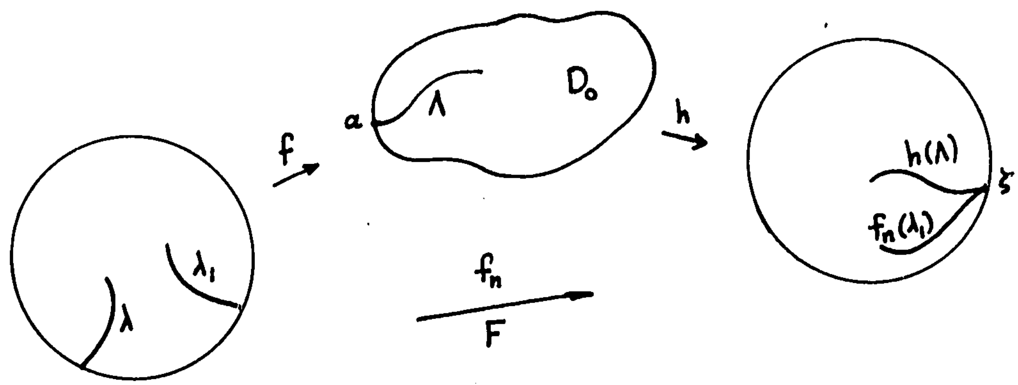


Figure 16.

$\{|z| = 1\}$ which is carries by $\check{f}_n$ into an arc $\Lambda_1 \subset \{|w| < 1\}$ ending at $h(a)$. Since $g_n = h \circ \check{f}_n$, it follows that $a \in \mathcal{A}(g_n) \subset \mathcal{A}$.

If $a \in \partial D_o$, then $a \in$ (accessible ∂D_o), since f maps $\{|z| < 1\}$ onto D_o (5.4). There exists an arc $\lambda \subset \{|z| < 1\}$ending at a point of $\{|z| = 1\}$ which is carried



Figrue 17.

by f into an arc Λ in D_o tending to a. Now h carries Λ into an arc ending at a point $\zeta \in \{|w| = 1\}$. Thus F has the asymptotic value ζ on λ, so that $\zeta \in \bigcup \mathcal{A}(f_n) \cup B$. If $\zeta \in B$, then $a \in \mathcal{A}$ by the definition of B; if $\zeta \in \mathcal{A}(f_n)$, then there exists an arc λ_1 in $\{|z| < 1\}$ on which f_n has the asymptotic value ζ. But then $g_n = \check{h} \circ f_n$ must have the asymptotic value a on λ_1, so that $a \in \mathcal{A}(g_n) \subset \mathcal{A}$.

Thus, $\mathcal{A}(f) \subset \mathcal{A}$.

Next, $\mathcal{A} \subset \mathcal{A}(f)$. Let $a \in \mathcal{A}$; again there are two cases to consider.

If $a \in D_o$, then $h(a) \in E$, so $h(a) \in \mathfrak{A}(F)$. If λ is an arc in $\{|z| < 1\}$ ending at a point of $\{|z|=1\}$ on which F has the asymptotic value $h(a)$, then $h \circ F = f$ must have the asymptotic value a on λ. Thus $a \in \mathfrak{A}(f)$.

If $a \in \partial D_o$, then by hypothesis $a \in$ (accessible ∂D_o). If Λ is an arc in D_o tending to a, then $h(\Lambda)$ is an arc in $\{|w| < 1\}$ tending to a point $\zeta \in B$. Since $\zeta \in \mathfrak{A}(F)$, there exists an arc λ_1 in $\{|z| < 1\}$ tending to a point of $\{|z| = 1\}$ on which F has the asymptotic value ζ; but then $f = \check{h} \circ F$ must have the asymptotic value a on λ_1, hence $a \in \mathfrak{A}(f)$.

Thus, $\mathfrak{A} = \mathfrak{A}(f)$. This completes the proof of the first case.

The second case is

2. D_o is parabolic. In this case we don't have the advantageous conformal homeomorphism h of the first case. However, we can apply the Lemma, since in this case we do have the advantage of $\mathfrak{A} \cap \partial D_o = \emptyset$.

Let $\mathfrak{A}_n = \mathfrak{A} \cap \{|w| < n\}$; then $\mathfrak{A}_n$ is analytic and $\mathfrak{A} = \bigcup \mathfrak{A}_n$. By hypothesis (2), there exists a holomorphic function g_n mapping $\{|z| < 1\}$ into D_o, with $\mathfrak{A}(g_n) \subset \mathfrak{A}$, and carrying a Jordan region topologically onto $\{|w| \le n\}$. The Lemma implies the existence of a function f_n holomorphic in $\{|z| < 1\}$, with $\mathfrak{A}(f_n) = \mathfrak{A}(g_n) \cup \mathfrak{A}_n$, whose Riemann surface $\mathfrak{F}_n$ contains a schlicht sheet D_n over $\{|w| \le \frac{n}{2}\}$. Thus, for any integer n, $\mathfrak{F}_n$ is schlicht over $\{|w| \le \frac{1}{2}\}$. Let β be the segment $[-\frac{1}{2},-\frac{1}{4}]$ and γ the segment $[\frac{1}{4},\frac{1}{2}]$.

Cut $D_1 \subset \mathfrak{R}_1$ along β and the remaining $D_n \subset \mathfrak{R}_n$ along both β and γ. Now form a Riemann surface $\hat{\mathfrak{R}}$ by joining the $\mathfrak{R}_n$ along common cuts according to the following scheme: For n odd, join $\mathfrak{R}_n$ to $\mathfrak{R}_{n+1}$ along β so as to form first order branch points over $-\frac{1}{2}$ and $-\frac{1}{4}$; for n even, join $\mathfrak{R}_n$ to $\mathfrak{R}_{n+1}$ along γ so as to form first order branch points over $\frac{1}{4}$ and $\frac{1}{2}$.

It is evident that the asymptotic paths on $\hat{\mathfrak{R}}$ determine the same asymptotic values as do those asymptotic paths on the various $\mathfrak{R}_n$, namely $\bigcup \mathfrak{a}(f_n)$; but $\bigcup \mathfrak{a}(f_n) = \bigcup (\mathfrak{a}(g_n) \cup \mathfrak{a}_n) = \mathfrak{a}$.

Let $\mathfrak{R}$ be the universal covering surface of $\hat{\mathfrak{R}}$. Using (3.6), we see that $\mathfrak{R}$ determines the same asymptotic set as does $\hat{\mathfrak{R}}$, and hence the function f holomorphic in $\{|z| < 1\}$ determined by $\mathfrak{R}$ has $\mathfrak{a}(f) = \mathfrak{a}$.

This completes the proof of the theorem. $\square$

COROLLARY. <u>The characterization of the theorem is also a characterization of the asymptotic sets of normal holomorphic functions</u>.

Proof. This is implied by Theorem 2. $\square$

Remark. MacLane [8] gives an example of holomorphic function mapping $\{|z| < 1\}$ onto $\{|w| < \infty\}$ with $\infty \notin \mathfrak{a}(f)$. Hence the second case considered above may actually arise.

In closing, we mention the following conjecture.

CONJECTURE. <u>Theorem</u> 7 <u>holds with condition</u> (2) <u>replaced by</u>

(2′) <u>given compact</u> $K \subset D_o$, <u>there exists a domain</u> D_k <u>with the properties</u>

(d′) $K \subset D_k \subset D_o$

(b) <u>(accessible)</u> $\partial D_k \subset \mathfrak{a}$.

Example 1 shows that <u>some</u> condition (2) is necessary to characterize the asymptotic sets of holomorphic functions in $\{|z| < 1\}$. It is easy to see that (2′) implies (2), so the conjecture does give a <u>sufficient</u> condition.

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