Achievable Rates for Arbitrary Network Topologies with 'Cheap' Nodes

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Abstract — **In this paper, we derive achievable rates for arbitrary network topologies consisting of 'cheap' nodes. A node is labeled 'cheap' if its radio can only operate in TDD mode when transmitting and receiving in the same frequency band. Two main results are shown. The first result provides an achievable rate for the channel with either continuous alphabet or discrete channel. The second result provides the achievable rate for the Gaussian channel with average power constraint. The two results are applied to the case of Gaussian relay channel and concatenated channel with cheap nodes.**

I. Introduction

Most radios used in commercial wireless systems operate in TDD mode when the transmitting and receiving frequencies are the same. Although it is possible to build the RF radios that are able to send and receive in the same frequency band, design of such radios needs precise and expensive components. Thus, we assume that the nodes of the network can either send or receive at a given time or use of the network. We label such a node radio as a *cheap* radio and the corresponding node of the network as a *cheap* node. We introduced the problem of 'cheap' networks in our prior work [5, 6], and addressed the capacity for the relay and concatenated channels. In this paper, we extend our results to the networks with arbitrary topologies.

In this paper, we first derive achievable rates (lower bound on the capacity) for the networks when the channel probability function is known (Theorem 1), and for the Gaussian channel networks with power constraint (Theorem 2). Then, we present applications of the results for two cases, the Gaussian cheap relay channel and multi-hop network with cheap nodes.

The relay channel was first defined and studied by Van der Mullen [9] for the case of three terminals (or nodes) and later considered by Cover [1, 3]. Several authors have recently considered a more complex version of relay channel [4, 11, 13] but all of the results assume the use of 'expensive' radios, which can send and receive at the same time in the same frequency band. This assumption is the key difference between the previous works and the results presented in this paper.

Rest of the paper is organized as follows: In section II, we derive achievable rate for the networks with arbitrary topologies. In section III, we present the application of both Theorems 1 and 2 for the Gaussian cheap relay channel considering two different cases for the power constraints. In section IV we show an application of the results in a more involved case of the multi-hop relay network with cheap nodes. Finally, we conclude with some remarks in section V.

II. General Results

We consider a general multi-terminal network of senders and receivers. The network can be considered as a directed graph where each node of the network represents a potential sender or receiver (or both), and each link represents existence of a (one-way) channel between two nodes.

In accordance with the term *channel use* in a single discrete memoryless channel, we define *network use* which corresponds to the simultaneous one time use of each existing nodes of the network either as a sender or receiver. Also, in each network use, set of *active links* is defined as all links which are exiting from the set of sender nodes. With this definition, the above mentioned practical constraint translates to the fact that in each network use there is no active link that enters at a sender node.

We define operating state of the network as a valid partitioning of the nodes of the network into two sets of 'sender nodes' and 'receiver nodes' such that there is no active link that enters at a sender node. It is obvious that if the number of nodes in the network is finite then the number of operating states of the network of cheap nodes are finite.

Assume that there are N nodes and for every node i, $X^{(i)}$ is the transmitted variable and $Y^{(i)}$ is the received variable. Following the notation of [12] , the whole network channel is represented by the channel transition function or channel probability function (c.p.f.) as: $P(y^{(1)}, y^{(2)}, \ldots, y^{(N)}|\ x^{(1)}, x^{(2)}, \ldots, x^{(N)}|m)$ which is the conditional probability density function (or mass function in the case of discrete alphabet) in state m. The input symbol $X_k^{(i)}$ which is the transmitted signal from node i in time k (or k'th channel use) depends on $W^{(ij)}$ $\forall j \epsilon \{1, 2, \ldots, N\}$ which is the intended message from node i to node j, and also the past values of the received signal $Y^{(i)}$, i.e. $Y_1^{(i)}, Y_2^{(i)}, \ldots, Y_{k-1}^{(i)}$.

Without loss of generality, we assume that node 1 is the source node and node N is the final destination node. Now suppose that state of the network is a deterministic function for every network use k as m_k, $m_k \epsilon \{1, 2, \ldots, M\}$, where M is the number of possible states assuming finite number of states. For any state m, define $n_m(k)$ as the number of states which is equal to m in the first k network uses. Let $t_m = \lim_{k \longrightarrow 0} n_m(k)/k$ define the portion of the time that network have been used in state m as the total number of network use goes to infinity. Then, the following can be shown:

Theorem 1 *Consider a general network with finite number of states, M, for which the sequence m_k of the states* *of the network is known a priori to all nodes. If set of the information rates $\{R_{ij}^m, i, j \in \{1, 2, \dots, N\}\}$ from node i to node j are achievable in state m given previous sequence of the states, then the rate R given by:*

$$R \le \sum_{j=2}^{N} \sum_{m=1}^{M} \delta_{ij}^m R_{ij}^m \quad (1)$$

is achievable from node 1 to node N, provided that:

$$\sum_{j=1,(j\neq i)}^{N} \sum_{m=1}^{M} t_m \delta_{ji}^m R_{ji}^m \le \sum_{j=1,(j\neq i)}^{N} \sum_{m=1}^{M} t_m \delta_{ij}^m R_{ij}^m \quad (2)$$

for all the nodes $i \in \{2, 3, \dots, N-1\}$, where $\delta_{ij}^m = 1$ iff link from node i to node j is active in the state m, otherwise $\delta_{ij}^m = 0$.

Proof See [8].

Theorem 1 provides a general framework to find an achievable rate between a single source-destination pair of interest in an arbitrary cheap relay network. It can be used wherever the channel transition probability of the network is fixed given the state of the network as defined earlier. Thus, it is applicable to the network with continuous input alphabet (e.g. Gaussian) as well as the network with discrete input alphabet as long as the channel is memoryless and the channel transition probability is fixed for each state. In other words, it is applicable in the case of the Gaussian channel network or block fading channel network with perfect CSIT and CSIR when the average power constraint is given for each nodes per state. On the other hand the following theorem extends the result to the case where the average power constraint is given for the whole block of transmission, which is of more interest for the Gaussian channel network.

Theorem 2 *Consider a Gaussian channel network with arbitrary topology and finite number of states, M, for which the sequence m_k of the states of the network is known a priori to all nodes. Suppose that the average power constraint per node is given as $P_0, P_1, \dots, P_{N-1}$ for the nodes $1, 2, \dots, N-1$. If set of the information rates $\{R_{ij}^m, i, j \in \{1, 2, \dots, N\}\}$ from node i to node j are achievable in state m given previous sequence of the states, then the following rate is achievable from node 1 to node N*

$$R \le \sum_{j=2}^{N} \sum_{m=1}^{M} \delta_{ij}^m R_{ij}^m \quad (3)$$

provided that:

$$\sum_{j=1,(j\neq i)}^{N} \sum_{m=1}^{M} t_m \delta_{ji}^m R_{ji}^m \le \sum_{j=1,(j\neq i)}^{N} \sum_{m=1}^{M} t_m \delta_{ij}^m R_{ij}^m \quad (4)$$

for all the nodes $i \in \{2, 3, \dots, N-1\}$, where $\delta_{ij}^m = 1$ iff link from node i to node j is active in the state m, and

$$P_i = \sum_{j=1,(j\neq i)}^{N} \sum_{m=1}^{M} t_m \delta_{ij}^m P_{ij}^m \quad (5)$$

for all the nodes $i \in \{1, 2, \dots, N-1\}$, where $P_i^m = \sum_{j=1,(j\neq i)}^{N} P_{ij}^m$ is the average power constraint of node i in the state m.

Proof See [8].

Maximal achievable rate of the Theorem 1 is achieved using the optimum sequence of the network states and the corresponding values of the time sharing parameters $t_1, t_2, \dots, t_M$. On the other hand, the maximal achievable rate in the Theorem 2 is achieved using joint optimization of the same parameters and power allocations P_{ij}^m for all possible values of i, j, and m.

Although results of the Theorems 1 and 2 do not provide the best possible achievable rate in general and it is sometimes possible to find a better achievable rate for some specific network topology, these results provide a lower bound for the capacity of the single source-destination pair in the network. It is worth mentioning that for some special cases, the derived lower bound is the exact capacity of the network. In [6], capacity of the cascaded single link channels (multi-hop) has been derived. Although it is not shown in this paper, it can be easily verified that the achievable rate of the Theorem 1 is in fact the capacity of cascaded channels.

On the other hand, apart from a very few specific cases, the problem of finding the capacity or even better achievable rate is intractable and the derived achievable rate of the Theorems 1 and 2 are the best known results. In the following sections we discuss two cases: (i) the Gaussian cheap relay channel, where a better achievable rate has been derived by means of very sophisticated coding techniques which have allowed us to fully characterize the corresponding capacity, and (ii) the Multi-hop cheap relay network, where the capacity is hard to find while Theorem 2 can give a reasonably good achievable rate in this case.

It is interesting to note that the assumption of the fluid model for the information flow in the network is not right. At the first look, the linear combination of the rates in the statement of the Theorems 1, 2 may give the impression of treating the information as such. But, close examination of the achievable rates in the each states reveals that the partial shared knowledge of the information between the nodes due to the previous transmission provide the basis of the cooperation between the nodes in the next states. In other words, the cooperation among the relay nodes results in better rates across some cut-set of the network which is contradicting the fluid model for information flow. This point would become more clear in the next section where we give more specific examples.

III. Gaussian Cheap Relay Channel

Consider Gaussian relay channel of Figure 1, in which source node S wants to transmit information to the destination node D by using direct link between the node pair (S, D) as well as help of another relay node R (if it improves the achievable rate of transmission) by using link pairs (S, R) and (R, D). Furthermore assume that relay node R is a 'cheap' node and thus it cannot transmit and receive at the same time.

A cheap relay channel, Figure 1, consists of an input x_1, a relay output y_1, a relay sender x_2 (which depends only on the past values of y_1), and a channel output y. The channel is assumed to be memoryless. With the assumption of cheap relay node, there are two possible

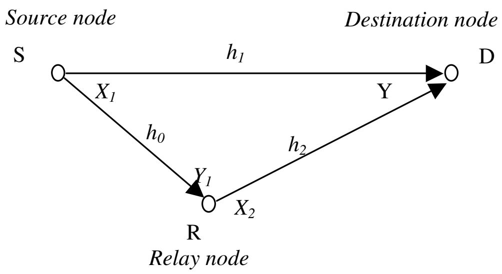

Figure 1: Gaussian Relay Channel

states of operation for the cheap relay channel. The dependency of the outputs on the inputs are as follows: In state m_1, relay node R acts as a receiver and thus the channel output is given by $y = h_1 x_1 + z$, and the relay output is given by $y_1 = h_0 x_1 + z_1$. In state m_2, relay node functions as a transmitter and the channel output is given by $y = h_1 x_1 + h_2 x_2 + z$, where h_0, h_1, and h_2 are channel losses and assumed to be constant, and $z \sim \mathrm{N}(0, N)$ and $z_1 \sim \mathrm{N}(0, N_1)$ are independent Gaussian noises. Suppose that we can arbitrarily choose between the states m_1 and m_2 in order to maximize the rate of information transfer between the source node S and destination node D. Suppose that in a total of n channel uses, the channel is used k times in the state m_1, and $n-k$ times in the state m_2. We assume the following three power constraints:(i) $\frac{1}{k}\sum_{i=1}^{k} x_{1i|m_1}^2 \leq P_0$, (ii) $\frac{1}{n-k}\sum_{i=1}^{n-k} x_{1i|m_2}^2 \leq P_1$, and (iii) $\frac{1}{n-k}\sum_{i=1}^{n-k} x_{2i|m_2}^2 \leq P_2$, where $x_{ji|m_k}$ is the i'th transmitted signal from node j while the channel is used in state m_k. The problem is to find the capacity of the channel between the sender S and receiver D.

Using Theorem 1 gives the following achievable rate for the channel if the channel is degraded:

$$R = \max_{0 \leq t, \alpha, \beta \leq 1} t(R_{11} + R_{01}) + (1-t)R_{12} \tag{6}$$

where $(1-t)R_{22} \leq tR_{01}$, and

$$\begin{aligned}
R_{11} &= \frac{1}{2}\log(1 + \frac{\gamma_1(1-\alpha)P_0}{1+\gamma_1\alpha P_0}) \\
R_{01} &= \frac{1}{2}\log(1+\gamma_0\alpha P_0) \\
R_{12} &\leq \frac{1}{2}\log(1+\gamma_1(1-\beta)P_1) \\
R_{12} + R_{22} &\leq \frac{1}{2}\log(1+\gamma_1 P_1 + \gamma_2 P_2 + 2\sqrt{\gamma_1\gamma_2\beta P_1 P_2})
\end{aligned}$$

and γ_i's, $i = 0, 1, 2$ are defined as: $\gamma_0 = \frac{|h_0|^2}{N_1}$, $\gamma_1 = \frac{|h_1|^2}{N}$, and $\gamma_2 = \frac{|h_2|^2}{N}$. The rates R_{01} and R_{11} are the achievable rate region of the broadcast channel in state m_1 and the rate R_{12} and R_{22} are the achievable rate region of the multiple access channel in state m_2.

The multiple access channel has been extensively studied and the achievable rate regions for the case of the independent sources, the correlated sources [2] and the hierarchical sources [10] have been derived. Also, the converse has been established for the independent message sources which results in the derivation of the capacity in this case. A result which may be of independent interest is that by proving the converse, we have shown that the rates R_{12} and R_{22} are the exact capacity region of multiple access channel where the node S knows the information of the node R (hierarchical case).

It is also well known that for the Gaussian degraded channel the rates R_{01} and R_{11} are exactly the capacity region. However, the capacity of the degraded Gaussian cheap relay channel is not given by the Equation (6), although the capacity region is known for both of the states. In fact, by using a more sophisticated coding technique we have shown that a higher rate than (6) is achievable for the Gaussian cheap relay channel [7] which allows us to fully determine the capacity of the degraded Gaussian cheap relay channel. Using the same notation and defined variables we have:

Theorem 3 *[7] Capacity of the Gaussian degraded cheap relay channel is given by:*

$$\begin{aligned}
C = \max_{t,\rho,\ 0\leq t,\rho\leq 1} \min\{&t\ \log(1+\gamma_0 P_0) + \\
&(1-t)\ \log(1+(1-\rho^2)\gamma_1 P_1),\ t\ \log(1+\gamma_1 P_0) + \\
&(1-t)\ \log(1+\gamma_1 P_1 + \gamma_2 P_2 + 2\rho\sqrt{\gamma_1\gamma_2 P_1 P_2})\}
\end{aligned} \tag{7}$$

The reason for achieving higher rate in the Theorem 3 is that the received signal in the state m_1 contains some extra information which can be deduced using joint decoding of all received signals.

In the case of the total average power constraint per nodes as $E[x_1^2] \leq P_1'$ and $E[x_2^2] \leq P_2'$ we can use the Theorem 2 and solve explicitly for t, which gives the achievable rate as:

$$R = \max_{0\leq\alpha,\beta\leq 1} \frac{R_{11}R_{22} + R_{01}R_{22} + R_{12}R_{01}}{R_{22} + R_{01}} \tag{8}$$

where the maximization is now subject to these power constraints:

$$P_1' = \frac{R_{22}P_0 + R_{01}P_1}{R_{22} + R_{01}} \tag{9}$$

$$P_2' = \frac{R_{01}P_2}{R_{22} + R_{01}} \tag{10}$$

IV. Multi-hop Gaussian Cheap Relay Channel

Consider the multi-hop Gaussian relay channel of Figure 2, in which source node S wants to transmit information to the destination node D with the help of two cheap relay nodes P and Q. It consists of an input x_0, two relay outputs y_1 and y_2, two relay sender x_1 and x_2 (which depends only on the past values of y_1 and y_2), and a channel output y. The channel is assumed to be memoryless. Since the relay nodes are cheap, there are three possible states of operation for this network. The dependency of the outputs on the inputs are as follows: In state m_1, both relay nodes P and Q act as a receiver and thus the output at node P is given by $y_1 = h_0 x_0 + z_1$, and the relay Q output is given by $y_2 = h_1 x_0 + z_2$. In state m_2, relay node P functions as a transmitter and the other relay works as a receiver, and the received signal is given by $y_2 = h_1 x_0 + h_2 x_1 + z_2$. In state m_3, relay node Q functions as a transmitter and the other relay works as a receiver. Then the received signal at node P

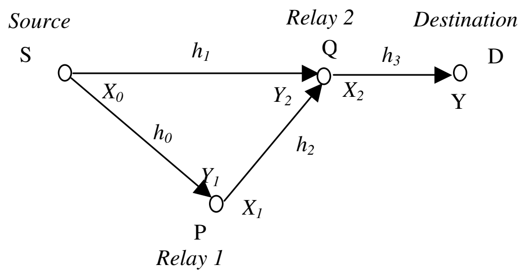

Figure 2: Multi-hop Gaussian Relay Channel

is given by $y_1 = h_0x_0 + z_1$, and the channel output y is given by $y = h_3x_2 + z$, where h_0, h_1, h_2, and h_3 are channel losses and assumed to be constant, and $z \sim N(0, N)$, $z_1 \sim N(0, N_1)$ and $z_2 \sim N(0, N_2)$ are independent Gaussian noises. Again, we suppose that we can arbitrarily choose between the states m_1, m_2, and m_3 in order to maximize the rate of information transfer between the source node S and destination node D. Let t_1, t_2, and t_3 represent the time proportion of being in the states m_1, m_2, and m_3, respectively.

In the case of the total average power constraint per nodes as $E[x_0^2] \leq P_1$, $E[x_1^2] \leq P_1$, and $E[x_2^2] \leq P_2$, we can use the Theorem 2 to derive the following achievable rate for the channel, if $N_2 \leq N_1$:

$$R = \max_{0 \leq t_1, t_2, t_3, \beta \leq 1} t_1(R_{11} + R_{01}) + t_2R_{12} + t_3R_{03} \quad (11)$$

Subject to three sets of constraints:
(i)

$$\begin{aligned} t_3R_{33} &\leq t_1R_{11} + t_2(R_{12} + R_{22}) \\ t_2R_{22} &\leq t_1R_{01} + t_3R_{03} \end{aligned}$$

(ii)

$$\begin{aligned} R_{11} &= \frac{1}{2}\log(1 + \frac{\gamma_1 P_{11}}{1 + \gamma_1 P_{01}}) \\ R_{01} &= \frac{1}{2}\log(1 + \gamma_0 P_{01}) \\ R_{12} &\leq \frac{1}{2}\log(1 + \gamma_1(1-\beta)P_{12}) \\ R_{12} + R_{22} &\leq \frac{1}{2}\log(1 + \gamma_1 P_{12} + \gamma_2 P_{22} + 2\sqrt{\gamma_1\gamma_2\beta P_{12}P_{22}}) \\ R_{03} &= \frac{1}{2}\log(1 + \gamma_0 P_{03}) \\ R_{33} &= \frac{1}{2}\log(1 + \gamma_3 P_{33}) \end{aligned}$$

(iii)

$$\begin{aligned} P_1 &= t_1(P_{01} + P_{11}) + t_2P_{12} + t_3P_{03} \\ P_2 &= t_2P_{22} \\ P_3 &= t_3P_{33} \end{aligned}$$

where γ_i's, $i = 0, 1, 2$ are defined as: $\gamma_0 = \frac{|h_0|^2}{N_1}$, $\gamma_1 = \frac{|h_1|^2}{N}$, and $\gamma_2 = \frac{|h_2|^2}{N}$.

The rates R_{01} and R_{11} are the achievable rate region of the broadcast channel in state m_1 and the rate R_{12} and R_{22} are the achievable rate region of the multiple access channel in state m_2 with hierarchical source structure and the rate R_{03} and R_{33} are the corresponding single link capacities.

V. Concluding Remarks

We presented achievable rates for arbitrary network topologies with cheap nodes. Since the actual capacity is known for very limited set of network topologies, the results presented in this paper are important to obtain useful lower bounds to actual capacity for many cases of practical interest. Application of the new results were presented in the context of relay and multi-hop relay channels. It is also possible to show that the achievable rate in the Theorem 1 is good enough to find the capacity of multi-hop network with cheap nodes [6, 8].

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