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Wavelet Analysis of Fractional Brownian Motion in Multifractal Time

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Résumé – Nous tudions le *mouvement Brownien fractionnaire en temps multifractal*, un modle de processus multifractal propos recemment dans le cadre de l'tude de sries financires. Notre intrt porte sur les proprits statistiques des coefficients d'ondelette issus de la dcomposition de ces processus. Parmi ces proprits nous nous intressons particulirement aux corrlations rsiduelles (longue dpendance), la stationnarit, qui sont les composantes essentielles permettant de caractriser les performances statistiques des estimateurs de spectre multifractal, construits partir de transformes en ondelettes.

Abstract – We study *fractional Brownian motions in multifractal time*, a model for multifractal processes proposed recently in the context of economics. Our interest focuses on the statistical properties of the wavelet decomposition of these processes, such as residual correlations (LRD) and stationarity, which are instrumental towards computing the statistics of wavelet-based estimators of the multifractal spectrum.

1 Introduction

Fractional Brownian motion (fBm) has for long served as the archetype of a process with long range dependence (LRD). At the same time, positive increment processes such as multiplicative cascades have proven amenable models for processes with underlying multifractal structures. For both processes, wavelets have played a key role in their analysis and synthesis (see [1, 4, 5, 14] and [2, 7, 10]). Combining these two classes of processes in the so-called *fractional Brownian motions in multifractal time* (BM(MT)) a novel class of processes was introduced in [9] which are versatile enough to enable modeling of LRD and multifractal scaling independently (compare [11]).

In this pioneering study, we decompose BM(MT) onto an orthogonal wavelet basis, and show that under mild conditions the correlation of wavelet coefficients decay fast (they are even zero in some cases), despite a strong dependence structure that underlies the process. It is then straightforward to follow [12] and compute the statistics of the wavelet-based estimator of the multifractal spectrum of BM(MT) [2, 7, 11].

2 Background

2.1 Wavelets and local regularity

The discrete (orthogonal) wavelet decomposition of a signal x is defined as the following inner product [3]:

$$c_{j,k} \;=\; 2^{-j} \int \psi^*(2^{-j}t - k)\, x(t)\, dt, \quad (j,k) \in \mathbb{Z}^2 \qquad (1)$$

where ψ is the so-called mother wavelet. An *admissible* wavelet must satisfy

$$\int t^r\, \psi(t)\, dt = 0, \quad r = 0, \ldots, \mathcal{R} - 1. \qquad (2)$$

The parameter $\mathcal{R} \geq 1$, is called the cancellation order of ψ and relates to the regularity of the wavelet.

Processes with local singularities (cusps, ridges, edges, chirps, etc. . .) appear in many fields of endeavor [2, 7, 8, 10]. The singularity behavior of a process $x(t)$ at time t may be characterized by the singularity exponent $\alpha(t)$ defined as the largest α such that

$$|d_{j,k}| = O(2^{j\alpha}) \qquad \text{as } k2^j \to t. \qquad (3)$$

It is shown in [5, 6, 7] that this notion $\alpha(t)$ is closely related to the Hölder regularity of x at t. Multifractal spectra quantify (geometrically or statistically) the occurrence of $\alpha(t) = \alpha$ in a multiplicative process $x(t)$. The multifractal formalism allows to compute the multifractal spectrum from the power law decay of the moments of the wavelet coefficients across scale.

2.2 Fractional Brownian motion

Fractional Brownian motion (fBm) B_H is the unique process that :

1. is Gaussian,
2. is statistically self-similar with parameter $0 < H < 1$, i.e.

$$B_H(\lambda t) \stackrel{d}{=} \lambda^H B_H(t), \qquad (4)$$

3. and has stationary increments $G_H(t) = B_H(t+\Delta) - B_H(t)$.

Statistical self-similarity implies that the fBm is itself a non stationary process. Its covariance function reads as

$$\mathbb{E}B_H(t)B_H(s) = \frac{\sigma^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \quad (5)$$

Note that fBm is a non-stationary process, but that its increments are stationary. For $H > 1/2$, the autocorrelation function of the increments process $G_H(t)$ decays very slowly,

$$\mathbb{E}[G_H(t)G_H(t+\tau)] \approx \tau^{2H-2}, \; \tau \gg \Delta$$

with $2H-2 > -1$ which was termed *long range dependence* (LRD).

Regarding its multifractal properties let us note that fBm $B_H(t)$ has a local Hölder exponent $\alpha(t) = H$, $\forall t$. In other words, fBm is a *monofractal* process. Mainly for this reason, fBm fails to provide sufficient flexibility as a model for many real world measurements, where one typically sees the local regularity $\alpha(t)$ changing erratically with time.

2.3 Multifractal cascades

The most well known processes with truly multifractal properties, i.e. erratically changing $\alpha(t)$, are the random cascades. The *Binomial cascades*, a particular example, is most easily defined using a binary tree structure. Given identically distributed random variables μ_k^i ($k = 0, \ldots 2^{-j}-1$, $i = 0, 1, \ldots$) define the random measure μ on dyadic intervals by

$$\mu([k_j 2^j, (k_j+1)2^j]) = \mu_{k_0}^0 \cdot \mu_{k_1}^1 \cdots \mu_{k_j}^j. \quad (6)$$

Here, given k_j we set $k_{i-1} = k_i \operatorname{div} 2$ ($i = 1, \ldots, j$). Also, we assume that $\mu_{2k}^{i+1} + \mu_{2k+1}^{i+1} = 1$ (conservation of mass) and that the μ_k^i are otherwise independent.

Finally, let the *cascade process* be defined by

$$\mathcal{M}(t) := \int_0^t d\mu.$$

Such processes $\mathcal{M}(t)$ are well known to be true multifractals. In addition, they possess a rescaling property similar to fBm: for $0 \leq s < t \leq 1$

$$\mathcal{M}(2^j(k+t)) - \mathcal{M}(2^j(k+s)) \stackrel{d}{=} W_j\left(\mathcal{M}(t) - \mathcal{M}(s)\right). \quad (7)$$

Comparing with (4) note that here W_j is a random variable. Its distribution depends only on the scale j, e.g. for the Binomial cascade

$$W_j \stackrel{d}{=} \mu_{k_0}^0 \cdot \mu_{k_1}^1 \cdots \mu_{k_j}^j. \quad (8)$$

More elaborate multiplicative cascades which have been discovered recently [13] have additional nice properties such as distributions of increments $\mathcal{M}(t) - \mathcal{M}(s)$ depending only on $|t-s|$ and scaling of moments, i.e. given q there is a number $T(q)$ such that:

$$\mathbb{E}|\mathcal{M}(t) - \mathcal{M}(s)|^q = |t-s|^{T(q)}. \quad (9)$$

Note that (9) holds for the Binomial cascades for $q = 1$ with $T(1) = 1$ due to the conservation of mass.

2.4 Multifractional motion

Mono-fractal processes like fBm are too elementary to serve as models for multifractal processes. On the other hand, multiplicative cascades, albeit providing us with rich multifractal models, may be inappropriate for real world problems due to their approximately log-normal marginals. In [9, 11] a broad class of *multifractal motions* is proposed based on *multifractal time warping*:

$$\mathcal{B}(t) \; := \; B_H(\mathcal{M}(t)). \quad (10)$$

Such processes has nice statistical as well as rich multifractal properties as is shown in [11]. For instance, using the covariance structure (5) of the underlying fBm, we find:

$$\begin{aligned} &\mathbb{E}\mathcal{B}(t)\mathcal{B}(s) \; = \quad (11)\\ &\mathbb{E}\left\{\mathbb{E}\left[B_H(\mathcal{M}(t))B_H(\mathcal{M}(s))\,|\mathcal{M}(t), \mathcal{M}(s)\right]\right\} = \\ &\frac{\sigma^2}{2}\mathbb{E}\left[|\mathcal{M}(t)|^{2H} + |\mathcal{M}(s)|^{2H} - |\mathcal{M}(t) - \mathcal{M}(s)|^{2H}\right]. \end{aligned}$$

Note also the special case $H = 1/2$ where $\mathbb{E}[\mathcal{M}(t) - \mathcal{M}(s)] = t - s$ (compare (9) with $q = 1$) yields

$$\mathbb{E}\mathcal{B}(t)\mathcal{B}(s) = \sigma^2 \min(\mathcal{M}(t), \mathcal{M}(s))$$

similar to the case of plain fBm. The increment process at given lag Δ, $\mathcal{G}(k) := \mathcal{B}((k+1)\Delta) - \mathcal{B}(k\Delta)$ is then decorrelated, but nevertheless dependent.

For $H > 1/2$, $\mathcal{B}$ exhibits long range dependence analogously to fBm (compare (5) and (11)).

3 Statistics

3.1 Identical distributions

Assume that the wavelet ψ is compactly supported, say, for simplicity, that it is zero outside the interval $[0, 1]$. Assume furthermore that $\mathcal{M}$ satisfies (7). Then, at each scale j, the wavelet coefficients $c_{j,k}$ are identically distributed according to:

$$c_{j,k} \; \stackrel{d}{=} \; (W_j)^H \, c_{0,0}, \quad (12)$$

where W_j is the random variable appearing in (7).

Proof.
Noting first that $\psi(2^{-j}t - k)$ is zero outside $[k2^j, (k+1)2^j]$, applying then a change of variable we find

$$c_{j,k} \;\; = \;\; \int_0^1 \psi(t) B_H(\mathcal{M}(2^j(t+k)))\, dt.$$

Next, we use the admissibility condition $\int \psi = 0$ to get

$$c_{j,k} \; = \; \int_0^1 \psi(t)(B_H(\mathcal{M}(2^j(t+k))) - B_H(\mathcal{M}(2^j k)))\, dt.$$

Finally, combining the property (7) of the cascade $\mathcal{M}$, and the self-similarity (4) of the fBm B_H, we get:

$$c_{j,k} \; \stackrel{d}{=} \; \int_0^1 \psi(t) B_H(W_j \mathcal{M}(t))\, dt \; \stackrel{d}{=} \; (W_j)^H \, c_{0,0}.$$

3.2 Wide sense stationarity

If the multiplicative cascade $\mathcal{M}(t)$ has the property (9) with $q = 2H$ then, for any admissible wavelet, the residual correlation of wavelet coefficients reads at each scale j:

$$\begin{aligned} &\mathbb{E}c_{j,k}c_{j,k'} \; = \quad (13)\\ &-\frac{\sigma^2}{2}2^{-j}\int \psi(t) \int |s|^{T(2H)} \psi(t + 2^{-j}s - (k-k'))ds\, dt. \end{aligned}$$

This expression depends on the positions k and k' only in terms of their difference. Consequently, within each scale j, the series of wavelet coefficients $c_{j,k}$ is wide sense stationary.

As a matter of fact, stationarity of increments of $\mathcal{M}$ is sufficient to conclude wide sense stationarity.

Proof.
Exchanging order of expectation and integrals

$$\mathbb{E}c_{j,k}c_{j,k'} = 2^{-2j} \int\int \psi_{j,k}(t)\, \psi_{j,k'}(t')\, \mathbb{E}\mathcal{B}(t)\mathcal{B}(t')\, dt dt'.$$

Using the covariance structure (11) of $\mathcal{B}$ and the admissibility condition of the wavelet ($\int \psi_{j,k} = 0$), the residual autocorrelation function $\mathbb{E}c_{j,k}c_{j,k'}$ simplifies to:

$$-\frac{\sigma^2}{2} 2^{-2j} \int\int \psi_{j,k}(t)\, \psi_{j,k'}(t')\, \mathbb{E}[|\mathcal{M}(t) - \mathcal{M}(t')|]^{2H}\, dt\, dt'.$$

Now, using property (9) with $q = 2H$, and making obvious changes of variables, we get the claimed result (13).

3.3 Fast decay of the correlation

As was mentioned above, $\mathcal{B}$ exhibits LRD which is unfavorable from a statistical point of view for the direct estimation of the local regularity via increments. Using instead the wavelet decomposition for this task (compare (3)) we may profit from its decorrelation property – which we are about to establish now – similarly as was done for the estimation of H in the case of fBm [1, 5, 6].

Under conditions suitable to obtain (13) the covariance of the wavelet coefficients across scale decays as

$$\mathbb{E}c_{j,k}c_{j,k'} \approx O(|k - k'|^{T(2H)-2\mathcal{R}}), \quad |k - k'| \to \infty. \quad (14)$$

In words, the wavelet transform is able to remove the LRD present in the process $\mathcal{B}$ meaning that $T(2H) - 2\mathcal{R}$ is smaller than -1 (no LRD) for large enough $\mathcal{R}$ while the correlation of the increments of $\mathcal{B}$ decay with exponent $T(2H) - 2$ which is larger than -1 (LRD).

Proof.
Returning to (13), we first apply the change of variable $2^{-j}s = \tau$ and identify $\gamma_\psi(\tau) = \int \psi(t)\psi(t + \tau)dt$, the autocorrelation function of the wavelet ψ. Thus, we obtain

$$\mathbb{E}c_{j,k}c_{j,k'} = -\frac{\sigma}{2} 2^{jT(2H)} \int |\tau|^{T(2H)} \gamma_\psi(\tau - (k - k'))\, d\tau. \quad (15)$$

Next, we apply Parseval's formula to write this last expression in the frequency domain as

$$\mathbb{E}c_{j,k}c_{j,k'} = -\frac{\sigma^2}{2} 2^{jT(2H)} \int \frac{|\Psi(\nu)|^2}{|\nu|^{T(2H)+1}}\, e^{-i2\pi\nu(k-k')} d\nu, . \quad (16)$$

Here, Ψ denotes the Fourier transform of ψ. In the limit $\nu \to 0$ the regularity condition of ψ implies that $|\Psi(\nu)|^2 \approx |\nu|^{2\mathcal{R}}$. Estimating the discrete Fourier transform in (16) leads finally to (14).

Let us emphasize the particular case of a Wiener process ($H = 1/2$), and the Haar system. First, for $q = 2H = 1$ (9) holds with $T(q) = T(2H) = 1$ even for the Binomial cascade as mentioned before. The above expression (15) reads

$$\mathbb{E}c_{j,k}c_{j,k'} = -\frac{\sigma^2}{2} 2^{j} \int |\tau + (k - k')|\, \gamma_\psi(\tau)\, d\tau. \quad (17)$$

Furthermore, for the Haar wavelet γ_ψ can be calculated explicitly

$$\gamma_\psi(\tau) \begin{cases} = -3|\tau| + 1 & 0 \le |\tau| < 1/2 \\ = |\tau| - 1 & 1/2 \le |\tau| < 1 \\ = 0 & |\tau| \ge 1, \end{cases}$$

which allows for a tedious but straightforward calculation leading to

$$\mathbb{E}c_{j,k}c_{j,k'} \begin{cases} = \frac{\sigma^2}{12} & |k - k'| = 0 \\ = 0 & |k - k'| \ne 0. \end{cases}$$

In other words, the wavelet coefficients of $\mathcal{B}$ are then strictly decorrelated. So, conditioned on knowing $\mathcal{M}$ they actually become independent Gaussian variables. Figure 1 shows, using a Haar system, the wavelet coefficients auto-correlation function estimated on an experimental data set. The resulting sharp structure is in perfect agreement with the theory. Based on this decorrelation the multifractal spectrum of $\mathcal{B}$ has been estimated in Figure 2.

4 Conclusion

Fractional Brownian motion (fBm) and Binomial cascades have been instrumental as archetypes of processes with LRD, respectively monotonous multifractals. Similarly, *fBm in multifractal time* is liable to play a key role in the study of oscillating multifractal processes (estimation, modeling and synthesis). Here, we proved stationarity and fast decorrelation of the wavelet decomposition of such signals. Based on these properties the statistical properties of the wavelet-based estimators of the multifractal spectrum can now be developed following along the lines of [12].

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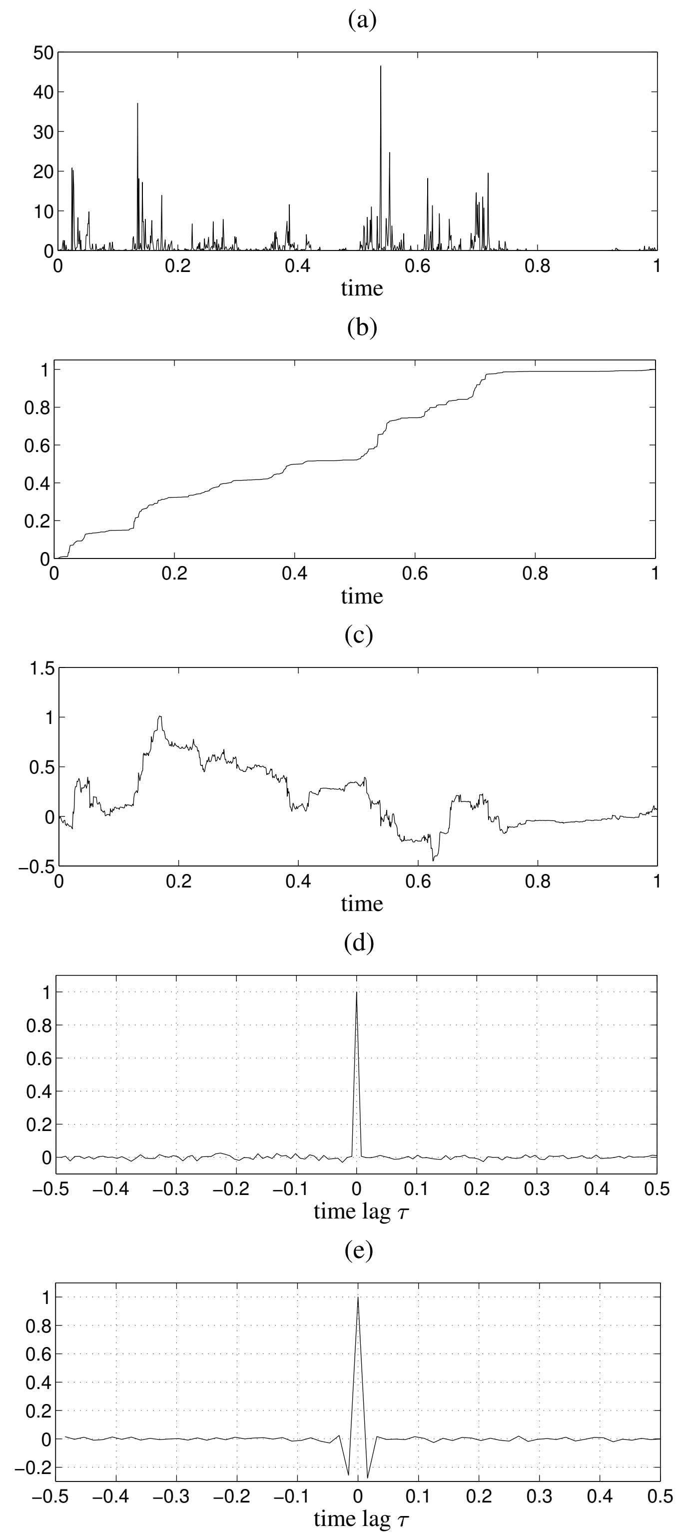

FIG. 1: *(a) Increments of one realization of a Binomial cascade $\mathcal{M}((k+1)2^j) - \mathcal{M}(k2^j) = \mu([k2^j,(k+1)2^j])$, $(k = 0, \dots 2^{-j}-1)$. (b) Realization of a Binomial cascade $\mathcal{M}(k2^j)$, $(k = 0, \dots 2^{-j}-1)$. (c) One realization of ordinary Brownian motion warped with the realization of (b), i.e. $\mathcal{B}(k2^j) = B_{1/2}(\mathcal{M}(k2^j))$ $(k = 0, \dots 2^{-j}-1)$. (d) Empirical residual correlation of the wavelet decomposition of $\mathcal{B}(t)$ using a Haar wavelet. (e) Empirical residual correlation of the wavelet decomposition of $\mathcal{B}(t)$ using a Daubechies wavelet with regularity* 2.

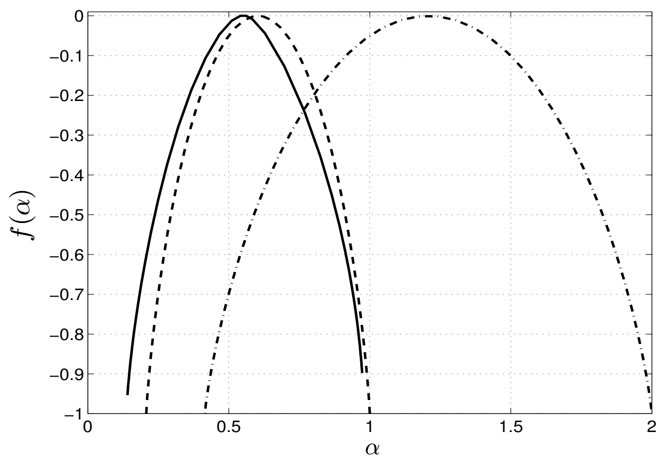

FIG. 2: *Multifractal analysis of $\mathcal{B}(t)$ displayed in figure 1. Dot-dashed: theoretical multifractal spectrum of the warp time $\mathcal{M}$ which is displayed in figure 1(b) – dashed: theoretical multifractal spectrum of the process $\mathcal{B}(t)$ itself which is displayed in figure 1(c) – solid: wavelet-based estimate of the multifractal spectrum of $\mathcal{B}(t)$. The spectra of warp time $\mathcal{M}$ and warped process $\mathcal{B}$ are related by $f_{\mathcal{B}}(\alpha) = f_{\mathcal{M}}(\alpha/H)$ (see [11]).*

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