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Discrete Morse Theory and the Geometry of Nonpositively Curved Simplicial Complexes

by

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ABSTRACT

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Understanding the conditions under which a simplicial complex collapses is a central issue in many problems in topology and combinatorics. Let K be a simplicial complex endowed with the piecewise Euclidean geometry given by declaring edges to have unit length, and satisfying the property that every 2-simplex is a face of at most two 3-simplices in K. Our main theorem is that if $|K|$ is nonpositively curved (in the sense of CAT(0)) then K simplicially collapses to a point. The main tool used in the proof is Forman's discrete Morse theory (see section 2.2), a combinatorial version of the classical smooth theory. A key ingredient in our proof is a combinatorial analog of the fact that a minimal surface in $\mathbb{R}^3$ has nonpositive Gauss curvature (see theorem 28). We also investigate another combinatorial question related to curvature. We prove a combinatorial isoperimetric inequality by finding an exact answer for the largest possible number of interior vertices in a triangulated n-gon satisfying the property that every interior vertex has degree at least six.

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Contents

Chapter 1

Introduction

One of the fundamental problems in mathematics is understanding the relationship between geometry and topology. In this thesis we investigate the relationship between these two fields in a combinatorial setting. There are many questions in geometry, topology, and other areas for which a combinatorial approach is more suitable than a continuous one. Understanding the relationship between geometry and topology from a combinatorial point of view is a powerful vantage point from which to approach many problems in a new light. One important classical link between geometry and topology is the Hadamard theorem, which states that a complete, simply connected, nonpositively curved manifold is contractible [Do]. The main result of this thesis is to establish analogous results in a combinatorial setting. In addition to being of significant independent interest, these results provide a foundation from which to further study the connections between combinatorial geometry and combinatorial topology.

In the late 1930s, J.H.C. Whitehead introduced the definition of simplicial collapse in an attempt to formulate homotopy theory in a purely combinatorial way [Wh]. Simplicial collapsibility is a combinatorial analog of contractibility for smooth spaces, but the two ideas are not equivalent. For example, every simplicial complex which collapses is contractible. However, the converse is not true: while the 3-ball is contractible, there are triangulated 3-balls that do not simplicially collapse to a point. For example, see example 3 in [Bi]. Understanding when a simplicial complex collapses is a central issue in a number of problems in topology and combinatorics, including the Poincaré conjecture. For example, if M is a combinatorial n-manifold with boundary which simplicially collapses to a point, Whitehead's theorem on regular neighborhoods then implies that M is combinatorially equivalent (and hence homeomorphic) to an n-ball [Wh].

In the study of combinatorial geometry, one has to make sense of the word curvature in a combinatorial space. The notion of nonpositive curvature we use is that given by the CAT(0) inequality, first introduced by A.D. Alexandrov [Al] and recently applied by M. Gromov to the study of hyperbolic groups [Gr]. A geodesic metric space is said to be CAT(0) if geodesic triangles are thinner than comparison triangles in Euclidean space, in the same way that triangles in hyperbolic space are thinner than in Euclidean space. CAT(0) spaces are necessarily contractible, and hence simply connected.

Our goal is to find geometric conditions that imply that a simplicial complex collapses. One set of geometric conditions is provided by Chillingworth in [Chi], where he proves that a triangulated 3-ball, embedded rectilinearly as a convex subset of $\mathbb{R}^n$, simplicially collapses. Here, we take a different approach and present a combinatorial analog of Hadamard's theorem.

The main tool we use in the proof is combinatorial Morse theory [Fo1], a discrete analog of classical, smooth Morse theory. In smooth Morse theory, one assigns a smooth function to a smooth manifold, and we know that level submanifolds deformation retract onto lower level submanifolds as long as one does not pass through a critical value. In discrete Morse theory, one assigns a real number to each simplex according to certain rules. In this case level subcomplexes simplicially collapse (and hence deformation retract) onto lower level subcomplexes as long as one does not pass through a critical value. (All of this is explained in section 2.2.) This makes discrete Morse theory a convenient tool for questions about simplicial collapse.

A discrete Morse function models a smooth Morse function in the sense that a noncritical simplex has a unique direction in which to "flow", while a critical simplex does not. Combinatorial versions of the main theorems of smooth Morse theory relate the topology of the simplicial complex to the critical points of the discrete Morse function. While the theories parallel each other closely, the lack of any smoothness requirement in combinatorial Morse theory makes it suitable for solving a different array of problems. Discrete Morse theory has been used to analyze a number of interesting questions in topology, graph theory, combinatorics, and complexity theory. For references, see [Fo1]. In particular, the methods developed by Forman can be used to give a combinatorial proof of the Poincaré Conjecture in dimensions five and higher, along the lines of the Morse theoretic proof presented by Milnor in the smooth category. The Poincaré Conjecture in dimension three, one of the biggest open problems in topology, can be restated in terms of combinatorial Morse theory as, "Every combinatorial 3-manifold M with boundary which is a homotopy 3-ball has a triangulation that admits a discrete Morse function with exactly one critical point." (A combinatorial manifold with boundary which has a discrete Morse function with exactly one critical point simplicially collapses. By Whitehead's theorem, M is a combinatorial 3-ball.) We will show that in a 3-complex where every 2-simplex is a face of at most two 3-simplices, to construct a Morse function with exactly one critical point, it is enough to show the complex has a discrete Morse function with exactly one critical vertex and no critical edges. A cancellation theorem allows us to cancel out in pairs the remaining critical 2- and 3-simplices.

The main idea of the proof is to fix a vertex v of our complex K and apply discrete Morse theory to the function "distance from v". (Note that this is essentially the main point of the proof of the Hadamard theorem.) The hypothesis that $|K|$ is CAT(0) is a restriction on the continuous distance function (resulting from the piecewise Euclidean structure on $|K|$. However, for simplicial collapse we are led to consider the combinatorial distance function on vertices obtained by only considering paths along edges. The critical issue in the proof of the theorem is understanding the relationship between these two distance functions. The more the edge lengths vary, the more tenuous the relationship between these two concepts of distance becomes. When the edges lengths are all unit length, we understand the relationship between the two notions of distance well enough to show the complex collapses.

Understanding the geometry of CAT(0) disks turns out to be the crucial step in the proof of the main theorem of the thesis. In section 3.1, we investigate the geometry of a CAT(0) triangulated disk, and in section 3.2 we outline a proof of how to simplicially collapse any triangulated disk D. One can present a proof which does not depend on any curvature hypothesis. However, we give a more complicated proof than needed, using what we have learned about the geometry of a CAT(0) disk. The proof we give serves as a preview for how the proof will be carried out in the three-dimensional case. The main point is that, given a simple closed curve in a disk formed by a union of edges, if e is an edge on the curve, it makes sense to speak of the 2-simplex incident to e which is "inside" the curve. In extending such ideas to a three-dimensional CAT(0) complex K, we need to have a notion of pointing "inside" a curve of which is a union of edges. The main result of section 3.3 is to show that in a three-dimensional complex K, any closed curve which is a union of edges bounds an immersed simplicial disk in K which is itself CAT(0). In fact we show that the simplicial disk of minimal area spanning the closed curve is CAT(0). This is a combinatorial analog of the fact that a minimal surface in $\mathbb{R}^3$ has nonpositive Gauss curvature (see theorem 27). The main theorem, that the three-dimensional CAT(0) complex K simplicially collapses, is proved in section 3.4.

In the final chapter we prove a combinatorial isoperimetric problem. The classical isoperimetric problem is to determine the largest area that can be enclosed in the plane by a curve with fixed perimeter. The calculus of variations evolved in part from attempts to solve this problem. For a history of the problem, see [HHM] and [Po]. In a triangulated disk, one notion of nonpositive curvature is to require that the sum of the angles around vertex be at least 2π. If the edges have unit length, this translates into the nice combinatorial description that every interior vertex has degree at least six. We use a combinatorial version of the Gauss-Bonnet theorem to find the maximum possible number of interior vertices of a triangulated n-gon, all of whose interior vertices have degree at least six. The question can also be viewed as a type of combinatorial packing problem. Isoperimetric inequalities play a substantial role in analysis of smooth spaces, and provide yet another important link between combinatorial geometry and combinatorial topology.

Chapter 2

Notation and Basic Definitions

2.1 Simplicial Complexes

A set $\{a_0, \ldots, a_n\}$ of points in $\mathbb{R}^N$ is said to be *geometrically independent* if for any real scalars t_i, the equations

$$\sum_{i=0}^{n} t_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} t_i a_i = \mathbf{0}$$

imply that $t_0 = t_1 = \cdots = t_n = 0$. It is easy to verify that $\{a_0, \ldots, a_n\}$ is a geometrically independent set if and only if the vectors $a_1 - a_0, \ldots, a_n - a_0$ are linearly independent vectors, as in linear algebra. A one-point set is always geometrically independent. Two distinct points in $\mathbb{R}^N$ form a geometrically independent set, as do three non-collinear points, four coplanar points, and so on.

Let $\{a_0, \ldots, a_n\}$ be a geometrically independent set in $\mathbb{R}^N$. We say that $a_0, \ldots, a_n$

span the set of points $x \in \mathbb{R}^N$ such that

$$x = \sum_{i=0}^{n} t_i a_i$$

for some $t_0, t_1, \ldots, t_n \geq 0$ and $\sum_{i=0}^{n} t_i = 1$. We define this set of points to be the *n-simplex* spanned by $\{a_0, \ldots, a_n\}$ and we define the *dimension* of σ to be n. Any simplex spanned by a subset of $\{a_0, \ldots, a_n\}$ is called a *face* of σ. We denote that σ is a face of β by writing $\sigma < \beta$. If σ^n is an n-simplex spanned by the n+1 vertices $v_0, v_1, \ldots, v_n$, we say that the face of σ spanned by the n vertices $v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ is the face *opposite* v_i. Similarly, v_i is called the vertex opposite the face spanned by $v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$. The faces of σ different from σ itself are called the *proper faces* of σ. Their union is the *boundary* of σ, denoted Bd σ. The *interior* of σ is defined by the equation Int $\sigma = \sigma -$ Bd σ. The set Int σ is called an *open simplex.*

A *simplicial complex* K in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$ such that every face of a simplex of K is in K and the intersection of any two simplices of K is a face of each of them. The *dimension* of K is defined to be the largest dimension of the simplices of K. If K has a finite number of simplices, we say that K is a *finite* simplicial complex. A *subcomplex* L of K is a subcollection of K that contains all faces of its elements. Let $\mathbf{K}^p$ denote the subcollection of K which consists of all p-simplices of K. Elements of K^0 are also called *vertices* of K and elements of K^1 are also called *edges* of K.

Let $|K|$ be the subset of $\mathbb{R}^N$ that is the union of the simplices of K. Giving each simplex its natural topology as a subspace of $\mathbb{R}^N$, we topologize $|K|$ by declaring a subset A of $|K|$ to be closed in $|K|$ if and only if $A \cap \sigma$ is closed in σ, for each σ in K. The space $|K|$ is called the *underlying space* of K. In general, the topology of $|K|$ is finer than the topology $|K|$ inherits as a subspace of $\mathbb{R}^N$. However, the topologies agree if K is finite.

For any vertex v of K, the *star* of v in K, denoted St v, is the union of the interiors of the simplices of K that have v as a vertex. The *closed star* of v in K, denoted $\overline{\text{St}}\ v$, is the closure of St v, or the union of all simplices of K that have v as a vertex. The set $\overline{\text{St}}\ v$-St v is called the *link* of v in K and is denoted Lk v.

A *triangulation* of a topological space X is a simplicial complex K and a homeomorphism $h : |K| \to X$. If there exists a triangulation of X, we say that X is a *polyhedron*.

Let K be a simplicial complex. Suppose that α is a p-dimensional simplex of K and α is not a proper face of any simplex in K. Suppose that β is a $(p-1)$-dimensional face of α but not of any other simplex in K. Then we say that K *simplicially collapses* onto $K - \{\alpha \cup \beta\}$.

Let K_1 and K_2 be simplicial complexes and let $\phi : K_1^0 \to K_2^0$ be a vertex map such that whenever the vertices $v_0, \ldots, v_n$ of K span a simplex of K_1, the vertices $f(v_0), \ldots, f(v_n)$ span a simplex of K_2. Then ϕ can be extended uniquely to a continuous map $|\phi| : |K_1| \to |K_2|$ such that

$$x = \sum_{i=0}^{n} t_i v_i \quad \Longrightarrow \quad |\phi|(x) = \sum_{i=0}^{n} t_i \phi(v_i).$$

The map $|\phi|$ is called the (linear) *simplicial map* induced by the the vertex map ϕ.

Let K be a simplicial complex. A *combinatorial path* in K from vertex v to vertex v' is a sequence

$$v = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1} = v'$$

such that

1. $v_i \neq v_{i+1}$ for $i = 1, \ldots, k$,

2. $v_i < e_i$ for $i = 1, \ldots, k$ and

3. $v_i < e_{i-1}$ for $i = 2, \ldots, k+1$.

We say that such a path has *length* k. If $v = v'$ we say the path is *closed*.

If there exists a combinatorial path from v to v' of length k, but there does not exist a combinatorial path from v to v' of length less than k, then any combinatorial path of length k is called a *combinatorial geodesic* from v to v'. Define the *combinatorial distance* $d_C(v, v')$ from v to v' to be the length of any combinatorial geodesic from v to v'.

We say that vertices v and v' are *neighbors* if $d_C(v, v') = 1$. We say that an interior vertex $v \in K$ is a *boundary neighbor* of K if there exists an exterior vertex $v' \in K$ such that $d_C(v, v') = 1$. We define the *degree* of a vertex v in K, denoted deg v, to be the number of distinct neighbors of v in K. When any confusion may result, we write $\deg_K v$ to denote the degree of the vertex v when considered as a vertex in the simplicial complex K.

A simplex σ of a simplicial complex K is an *exterior simplex* of K if $\{\sigma\} \subset \text{Bd } |K|$. A simplex σ of K is an interior simplex if it is not an exterior simplex. A triangulated disk with $n \geq 3$ distinct exterior vertices is called a *triangulated n-gon.*

Let X be a polyhedron and $\mathcal{U}$ an open covering of X. A triangulation (K, h) is said to be *finer* than $\mathcal{U}$ if for every vertex $v \in K$, there exists a $U \in \mathcal{U}$ such that $\text{St } h(v) \subset U$. A simplicial complex K is said to be finer than an open covering $\mathcal{U}$ of $|K|$ if for each vertex $v \in K$ there is a $U \in \mathcal{U}$ such that $\text{St } v \in U$.

Theorems 1 through 4 on the theory of simplicial approximations are taken from sections 1 through 4 in chapter 3 of Spanier's *Algebraic Topology* [Sp].

Theorem 1. *Let $\mathcal{U}$ be an open covering of a polyhedron X. Then there exists a triangulation (K, h) of X that is finer than $\mathcal{U}$.*

Let K_1 and K_2 be simplicial complexes and let $f : |K_1| \to |K_2|$ be continuous. If $|\phi| : |K_1| \to |K_2|$ is a simplicial map such that for all $x \in |K_1|$ and $\sigma \in K_2$, $f(x) \in \sigma$ implies $|\phi|(x) \in \sigma$, we say $|\phi|$ is a *simplicial approximation* to f. (In the next theorem, we write $|\phi| \simeq f$ to denote that $|\phi|$ is homotopic to f.)

Theorem 2. *Let $|\phi| : |K_1| \to |K_2|$ be a simplicial approximation to a map $f : |K_1| \to |K_2|$. Then $|\phi| \simeq f$.*

Theorem 3. *A map $|\phi| : |K_1| \to |K_2|$ is a simplicial approximation to $f : |K_1| \to |K_2|$ if and only if for every vertex $v \in K_1$, $f(St\ v) \subset St\ \phi(v)$.*

Theorem 4. *A map $f : |K_1| \to |K_2|$ admits simplicial approximations $|\phi| : |K_1| \to |K_2|$ if and only if K_1 is finer than the open covering $\{f^{-1}(St\ v) \mid v \text{ is a vertex of } K_2\}$.*

2.2 Discrete Morse Theory

In this section we present an overview of discrete Morse theory for simplicial complexes. All definitions and results in this section are from the paper Combinatorial Differential Topology and Geometry by Forman [Fo2]. For a nice introduction, see also [Fo1]. We begin by defining the concepts of a discrete Morse function and a critical point.

Definition 5. A function

$$f : K \to \mathbb{R}$$

is a *discrete Morse function* if for every $\alpha^{(p)} \in K$

1. $\#\{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} \leq 1$, and
2. $\#\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\} \leq 1$.

We see from the definition that, generally speaking, f assigns higher values to higher dimensional simplices, locally, with at most one exception at each simplex.

Definition 6. A simplex $\alpha^{(p)}$ is *critical* if

1. $\#\{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} = 0$, and
2. $\#\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\} = 0$.

We next define a combinatorial notion of a level set for a simplicial complex. If f is a discrete Morse function on a simplicial complex K, then for any real number

c, we define the *level subcomplex* $K(c)$ to be the subcomplex of K consisting of all simplices β such that $f\beta \leq c$, and all of their faces. That is,

$$K(c) = \bigcup_{f(\beta)\leq c} \bigcup_{\alpha\leq\beta} \alpha.$$

In analogy with smooth Morse theory, the following two theorems relate the topology of a simplicial complex K to the critical points of a discrete Morse function on K.

Theorem 7. *Suppose the interval $(a, b]$ contains no critical values of f. Then $K(a)$ is a deformation retract of $K(b)$. Moreover, $K(b)$ simplicially collapses onto $K(a)$.*

Theorem 8. *Suppose $\alpha^{(p)}$ is a critical simplex with $f(\alpha) \in (a, b]$, and there are no other critical simplices with values in $(a, b]$. Then $K(b)$ is (simple-)homotopy equivalent to*

$$K(a) \bigcup_{\dot{e}^{(p)}} e^{(p)}$$

where $e^{(p)}$ is a p-cell, and it is glued to $K(a)$ along its entire boundary $\dot{e}^{(p)}$.

Corollary 9. *Suppose K is a simplicial complex with a discrete Morse function. Then K is (simple-)homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical simplex of dimension p.*

For a simplicial complex with a discrete Morse function, let m_p denote the number of critical simplices of dimension p. Let $\mathbb{F}$ be any field, and $b_p = \dim H_p(K, \mathbb{F})$ the p^{th} Betti number with respect to $\mathbb{F}$. Then we have the following inequalities.

Corollary 10. *I. The Weak Morse Inequalities.*

1. For each $p = 0, 1, 2, \ldots, n$ *(where* n *is the dimension of* K*)*

$$m_p \geq b_p$$

2. $m_0 - m_1 + m_2 - \cdots + (-1)^n m_n = b_0 - b_1 + b_2 - \cdots + (-1)^n b_n = \chi(K)$.

II. The Strong Morse Inequalities.

For each $p = 0, 1, 2, \ldots, n, n+1$.

$$m_p - m_{p-1} + \cdots \pm m_0 \geq b_p - b_{p-1} + \cdots \pm b_0.$$

Discrete Morse theory is defined for general simplicial complexes. However, in the case that the complex is a combinatorial manifold, we can often say more. One example is the following theorem.

Corollary 11. *Suppose* K *is a combinatorial* n*-manifold with boundary with a Morse function with exactly one critical point. Then* K *is a combinatorial* n*-ball.*

This is a consequence of Whitehead's theorem on regular neighborhoods, which says that a collapsible combinatorial n-manifold with boundary is a combinatorial n-ball. If K is a complex with only one critical point, the critical point must be a vertex. Theorem 7 implies the complex collapses to that vertex, and Whitehhead's theorem then implies that K is a combinatorial n-ball.

We will now define a combinatorial notion of vector field, which we call a *gradient vector field*, associated to any discrete Morse function on a simplicial complex K. The gradient vector field is a function $V : K \to K \cup \{0\}$ defined as follows. If $\beta^{(p+1)} > \alpha^{(p)}$ are simplices that satisfy $f(\beta) \leq f(\alpha)$ then we set $V(\alpha) = \beta$. Define $V(\alpha)$ to be 0 for all simplices α for which there is no such β.

Let α and $\tilde{\alpha}$ be p-simplices. A *gradient path* from $\tilde{\alpha}$ to α is a sequence of simplices

$$\tilde{\alpha} = \alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \ldots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)} = \alpha$$

such that for each $i = 1, \ldots, r$, $f(\alpha_i) \geq f(\beta_i) > f(\alpha_{i+1})$. Equivalently, $V(\alpha_i) = \beta_i$, and $\beta_i > \alpha_{i+1} \neq \alpha_i$.

We will see that the gradient vector field V is often easier to work with than the actual Morse function. For this reason, it is useful to have a characterization of which vector fields are gradient vector fields of discrete Morse functions. We define a general discrete vector field and then give a necessary and sufficient condition for a discrete vector field to be the gradient vector field of a discrete Morse function.

Definition 12. A *discrete vector field* is any map

$$U : K \to K \cup \{0\}$$

satisfying for each $\alpha^{(p)}$

1. $U(\alpha) = 0$ or α is a codimension-one face of $U(\alpha)$.

2. If $\alpha^{(p)} \in \text{Image}(U)$ then $U(\alpha) = 0$.

3. If $\alpha^{(p)} \in \text{Image}(U)$ then there exists exactly one simplex $\gamma \in K$ with $U(\gamma) = \alpha$.

If U is a discrete vector field, we define a *U-path* to be a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \ldots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}$$

such that for each $i = 1, \ldots, r$, $\beta_i = U(\alpha_i)$ and $\beta_i > \alpha_{i+1} \neq \alpha_i$. We say such a path is a *non-trivial closed path* if $r \geq 0$ and $\alpha_0 = \alpha_{r+1}$.

Theorem 13. *A discrete vector field U is the gradient vector field of a discrete Morse function if and only if there are no non-trivial closed U-paths.*

The critical simplices of the discrete Morse function corresponding to a discrete vector field on K are precisely the simplices $\alpha \in K$ such that α is not in the image of U and $U(\alpha) = 0$.

In [Fo2] Forman shows how to develop a chain complex $\mathcal{K}$, called the *Morse complex*,

$$\mathcal{K}: \ 0 \longrightarrow \mathcal{K}_n \xrightarrow{\partial} \mathcal{K}_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{K}_1 \xrightarrow{\partial} \mathcal{K}_0 \longrightarrow 0$$

which has the same homology as the underlying space $|K|$, and from which one can obtain a more complete description of the relationship between the critical simplices of a discrete Morse function. The p^{th} chain group $\mathcal{K}_p$ is generated by the critical simplices of dimension p. If β is a critical p-simplex and α is a critical $(p-1)$-simplex, the value of $\partial\beta$ on α is the number of gradient paths (counted with orientation) from β to α. (For the proof of the main theorem we will not need to be concerned about the orientation of simplices.) For a complete description of the Morse complex, refer to [Fo2].

Finally, we note the following theorem which shows that, under certain conditions, we can simplify a discrete Morse function by "canceling" out critical simplices.

Theorem 14. *Suppose f is a discrete Morse function on K such that $\beta^{(p+1)}$ and $\alpha^{(p)}$ are critical, and there is exactly one gradient path from $\partial\beta$ to α. Then there is another Morse function g on K with the same critical simplices except that α and β are no longer critical. Moreover, the gradient vector field associated to g is equal to the gradient vector field associated to f except along the unique gradient path from $\partial\beta$ to α.*

2.3 CAT(0) Spaces

Let (X, d) be a metric space. The closed ball with center x and radius r is denoted by $B_r(x)$. A *path* in X is a continuous map $I : [0, 1] \to X$. A *geodesic* between two points x and y in X is a path $g : [0, 1] \to X$ such that $g(0) = x$, $g(1) = y$, and $d(g(s), g(t)) = |s - t|$ for all $s, t \in [0, 1]$. A *geodesic segment* in X is the subset of X that is the image of a geodesic. A *geodesic metric space* is a metric space in which every pair of points can be joined by a geodesic segment.

Suppose that $K \subseteq \mathbb{R}^N$ is a simplicial complex endowed with the piecewise Euclidean geometry given by declaring edges to have unit length. We obtain a metric on $|K|$ by taking the distance between two points x and y in $|K|$ to be the infimum over all paths in $|K|$ from x to y.

A *geodesic triangle* $\Delta = (x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points (vertices) x_1, x_2, and x_3 in X and a geodesic segment (edge) between each pair of vertices. A *comparison triangle* for the geodesic triangle $\Delta = (x_1, x_2, x_3)$ is a geodesic triangle $\Delta' = (x'_1, x'_2, x'_3)$ in $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(x'_i, x'_j)$ for all i, j. If a is a point on the geodesic segment of Δ from x_i to x_j, $i \neq j$, then the point a' on the comparison triangle Δ' satisfying $d_{\mathbb{R}^2}(x'_i, a') = d(x_i, a)$ and $d_{\mathbb{R}^2}(x'_j, a') = d(x_j, a)$ is called the point *corresponding* to a. X is a *CAT(0)* space if all geodesic triangles satisfy the following comparison axiom of Alexandrov and Toponogov:

Let a and b be any two points of Δ, and let a' and b' be the points of Δ' corresponding to a and b. Then $d(a, b) \leq d_{\mathbb{R}^2}(a', b')$.

Chapter 3

Collapsing Simplicial Complexes

Recall the definition of simplicial collapse from page 9: If K is a simplicial complex. α a p-dimensional simplex of K which is not a proper face of any simplex in K. and β a $(p-1)$-dimensional face of α but not of any other simplex in K. then we say that K *simplicially collapses* onto $K - \{\alpha \cup \beta\}$.

The goal of this chapter is to prove a theorem that gives geometric conditions which guarantee that a three-dimensional simplicial complex collapses to a point. We begin the chapter with a discussion of the geometry of CAT(0) disks and then prove that every triangulated disk collapses. The ideas of the proof will be applied to prove the main result of the chapter. which is that a three-dimensional nonpositively curved simplicial complex collapses to a point. More precisely. the theorem we prove is the following:

Theorem. *Let K be a finite 3-dimensional simplicial complex endowed with the piecewise Euclidean geometry given by declaring edges to have unit length, and satisfying* *the additional property that every 2-simplex of K is a face of at most two 3-simplices of K. If $|K|$ is CAT(0) then K simplicially collapses to a point.*

The chapter is divided into four sections. The first section focuses on the geometry of triangulated disks which are CAT(0). In the second section we will see how to put a discrete Morse function on a triangulated disk that shows it simplicially collapses to a vertex. As we will see in sections 3 and 4, a fundamental understanding of the problem in dimension two provides great insight into how to define a Morse function on the edges of a 3-dimensional complex. In section 3 we investigate the structure of CAT(0) simplicial disks immersed in a 3-complex. Section 4 is a proof of the theorem above.

3.1 The Geometry of CAT(0) Triangulated Disks

One of the most challenging steps in many topological problems is making the jump from two dimensions to three. The crucial point in the proof of the theorem stated at the beginning of the chapter makes use of two-dimensional CAT(0) disks immersed simplicially in the three-dimensional complex K. Therefore we begin the chapter with a section devoted to understanding the geometry of a CAT(0) triangulated disk. In particular we wish to understand the distance between vertices. measured along edges of the triangulation. We will see that the properties of such a distance function depend on the lengths of the edges of the triangulation.

Let D be triangulated disk. Choose a length for each edge satisfying the triangle

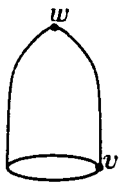

Figure 3.1: Distance from v is maximized in the interior of the positively curved disk.

inequality and endow D with the corresponding piecewise Euclidean metric. Then D is CAT(0) if and only if the sum of the angles around each interior vertex is at least 2π [Gr]. Choose a distinguished exterior vertex v of D, and define the combinatorial distance from each vertex to v to be the sum of the lengths of the edges in the shortest edge-path between them. Our goal is to study this distance function. In particular, we will show that if all edges are assigned the length 1 and the corresponding piecewise Euclidean metric is CAT(0), this distance function attains its maximum only on the boundary of the disk (see corollary 18). Note that without some assumption on curvature, it is not to be expected that a combinatorial distance function is maximized on the boundary of D (see figure 3.1).

In addition, if we do not put any restrictions on the edge lengths of D, then it is still not necessarily true that the combinatorial distance function is maximized on the boundary of D, as shown by the triangulated disk in figure 3.2, where edge lengths are taken to be their Euclidean distances as drawn in the plane. It is in the case of unit length edges that we understand the relationship between the continuous notion of distance in a CAT(0) space and the combinatorial notion of distance obtained by measuring along edges of the complex well enough to show that the complex collapses.

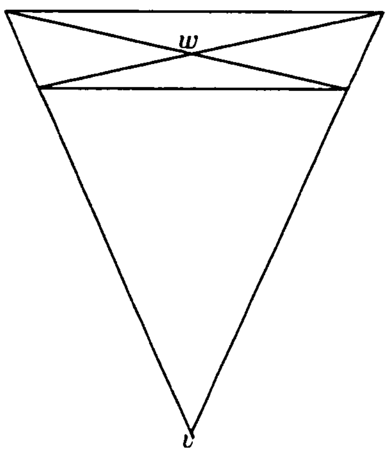

Figure 3.2: Vertex w is further from v than any other vertex, with distance measured along edges of the triangulation.

At the end of the section we give a preview of the proof of the collapsibility of CAT(0) 3-dimensional complexes by defining a Morse function on a triangulated disk where none of the simplices is critical except for one vertex. Theorem 7, one of the main theorems of discrete Morse theory, then implies that the disk simplicially collapses to a vertex. In dimension 2, the proof in fact works for any triangulated disk, regardless of the curvature.

We begin the section by presenting a well-known combinatorial formulation of the Gauss-Bonnet theorem relating curvature to the Euler characteristic of a triangulated disk.

Lemma 15. *Let D be a triangulated disk. Then the following combinatorial Gauss-*

Bonnet formula holds:

$$6 = \sum_{int\ v} (6 - deg\ v) + \sum_{ext\ v} (4 - deg\ v)$$

Proof. Let V be the number of vertices of D, V_{int} the number of interior vertices, V_{ext} the number of exterior vertices, E the number of edges, E_{int} the number of interior edges, E_{ext} the number of exterior edges, and F the number of 2-simplices of D. For any triangulation of the disk we have the following equations: $V - E + F = 1$ (Euler characteristic), $3F = 2E - E_{ext}, V_{ext} = E_{ext}$, and $\sum_{int\ v} \deg v + \sum_{ext\ v} \deg v = 2E$ for vertices $v \in D$ Using these equations we derive the formula:

$$\begin{aligned}
6 &= 6V - 6E + 6F \\
&= 6V - 6E + 6\left(\frac{2}{3}E - \frac{1}{3}E_{ext}\right) \\
&= 6V - 2E_{ext} - 2E \\
&= 6V - 2V_{ext} - \sum_{int\ v} \deg v - \sum_{ext\ v} \deg v \\
&= 6V_{int} + 4V_{ext} - \sum_{int\ v} \deg v - \sum_{ext\ v} \deg v \\
&= \sum_{int\ v} (6 - \deg v) + \sum_{ext\ v} (4 - \deg v).
\end{aligned}$$

□

Note that if every interior vertex of D has degree at least six, then lemma 15

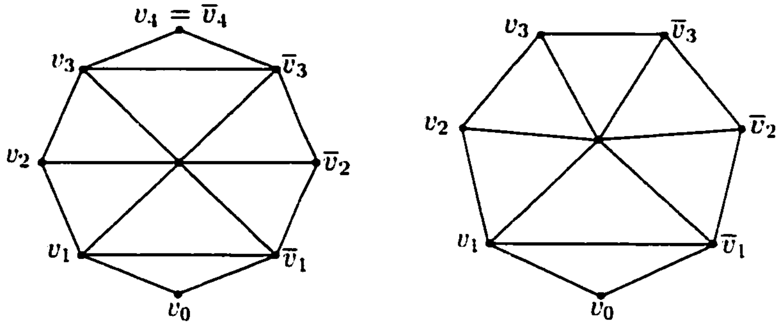

Figure 3.3: The disk on the left is a geodesic disk of type I. The disk on the right is a geodesic disk of type II.

implies

$$\sum_{ext\ v} (4 - \deg v) \geq 6. \tag{3.1}$$

This is a fact to which we will refer continually.

Let D be a triangulated disk whose exterior vertices are the vertices of the combinatorial geodesics $v_n, e_n, v_{n-1}, \ldots, v_1, e_1, v_0$ and $\overline{v}_n, \overline{e}_n, \overline{v}_{n-1}, \ldots, \overline{v}_1, \overline{e}_1, v_0$ such that $\deg v \geq 6$ for all interior vertices v. If $v_n = \overline{v}_n$ then D is called a *geodesic disk of type I*. If v_n is a neighbor of $\overline{v}_n$ then D is called a *geodesic disk of type II*.

Lemma 16. *If D is a geodesic disk of type I then there exists an exterior vertex v of D with $v \notin \{v_0, v_n\}$ satisfying $\deg v = 3$. If D is a geodesic disk of type II then there exists an exterior vertex v of D with $v \notin \{v_0, v_n, \overline{v}_n\}$ satisfying $\deg v = 3$.*

Proof. By definition, $\deg v \geq 2$ for all exterior vertices v of D. First we will show that $\deg v \geq 3$ for $v \in \{v_1, \overline{v}_1, \ldots, v_{n-1}, \overline{v}_{n-1}\}$. Suppose $\deg v_k = 2$ where $0 < k < n$. Then v_{k-1}, v_k, and v_{k+1} span a 2-simplex in D, which implies $d_C(v_{k+1}, v_{k-1}) = 1$. But $v_n, e_n, v_{n-1}, \ldots, v_1, e_1, v_0$ is a combinatorial geodesic so $d_C(v_{k+1}, v_{k-1}) = 2$, and we reach a contradiction. Thus deg $v_k \geq 3$. The argument is the same for $\overline{v}_k$.

Now consider a geodesic disk of either type I or type II. deg $v \geq 6$ for all interior vertices v so the combinatorial Gauss-Bonnet formula implies $\sum_{ext\ v}(4 - \deg v) \geq 6$. Then for a geodesic disk of type I we have

$$\begin{aligned}
6 &\leq \sum_{ext\ v}(4 - \deg v) \\
&= (4 - \deg v_0) + (4 - \deg v_n) + \sum_{ext\ v \neq v_0, v_n}(4 - \deg v) \\
&\leq 2 + 2 + \sum_{ext\ v \neq v_0, v_n}(4 - \deg v) \\
&= 4 + \sum_{ext\ v \neq v_0, v_n}(4 - \deg v)
\end{aligned}$$

Thus $\sum_{ext\ v \neq v_0, v_n}(4 - \deg v) \geq 2$, which means there must be an exterior vertex $v \notin \{v_0, v_n\}$ with deg $v < 4$. deg $v \geq 3$ so deg $v = 3$, as desired.

The proof is similar for a geodesic disk of type II. Observe that if deg $v_n = 2$ then v_{n-1} and $\overline{v}_n$ span an edge in D, so deg $\overline{v}_n \geq 3$. Thus it can't be true that both deg $v_n = 2$ and deg $\overline{v}_n = 2$. Without loss of generality we assume that deg $\overline{v}_n \geq 3$. Then

$$\begin{aligned}
6 &\leq \sum_{ext\ v} (4 - \deg v) \\
&= (4 - \deg v_0) + (4 - \deg v_n) + (4 - \deg \overline{v}_n) + \sum_{ext\ v \neq v_0, v_n, \overline{v}_n} (4 - \deg v) \\
&\leq 2 + 2 + 1 + \sum_{ext\ v \neq v_0, v_n, \overline{v}_n} (4 - \deg v) \\
&= 5 + \sum_{ext\ v \neq v_0, v_n, \overline{v}_n} (4 - \deg v)
\end{aligned}$$

Thus $\sum_{ext\ v \neq v_0, v_n, \overline{v}_n} (4 - \deg v) \geq 1$. which means D must have an exterior vertex $v \notin \{v_0, v_n, \overline{v}_n\}$ with $\deg v < 4$. Again, $\deg v \geq 3$ so $\deg v = 3$, as desired. $\square$

Let J be a simplicial complex whose underlying space is homeomorphic to $\mathbb{R}^2$ such that $\deg v \geq 6$ for all v in J. Let S be any subcomplex of J which is simply connected and whose exterior vertices are the (not necessarily distinct) vertices of the combinatorial geodesics $v_n, e_n, v_{n-1}, \ldots, v_1, e_1, v_0$ and $\overline{v}_n, \overline{e}_n, \overline{v}_{n-1}, \ldots, \overline{v}_1, \overline{e}_1, \overline{v}_0$ in J with $v_0 = \overline{v}_0$. If $v_n = \overline{v}_n$ the subcomplex is called a *string of pearls of type I*. If v_n is a neighbor of $\overline{v}_n$ the subcomplex is called a *string of pearls of type II*. Note that a geodesic disk of type I is a special case of a string of pearls of type I and a geodesic disk of type II is a special case of a string of pearls of type II.

Consider a string of pearls S either of type I or type II, with exterior vertices as described above. Suppose that the vertices $v_i, \overline{v}_i, v_j$ and $\overline{v}_j$ satisfy $i < j$, $v_i = \overline{v}_i$, and $v_k \neq \overline{v}_k$ for any $i < k < j$. The first case we consider is when $v_j = \overline{v}_j$. In this case, $v_j, e_j, v_{j-1}, \ldots, v_{i+1}, e_{i+1}, v_i = \overline{v}_i, \overline{e}_{i+1}, \overline{v}_{i+1}, \ldots, \overline{v}_{j-1}, \overline{e}_j, \overline{v}_j$ is a closed combinatorial path which bounds a disk D_{ij} in S. It is easy to verify that $v_j, e_j, v_{j-1}, \ldots, v_{i+1}, e_{i+1}, v_i$

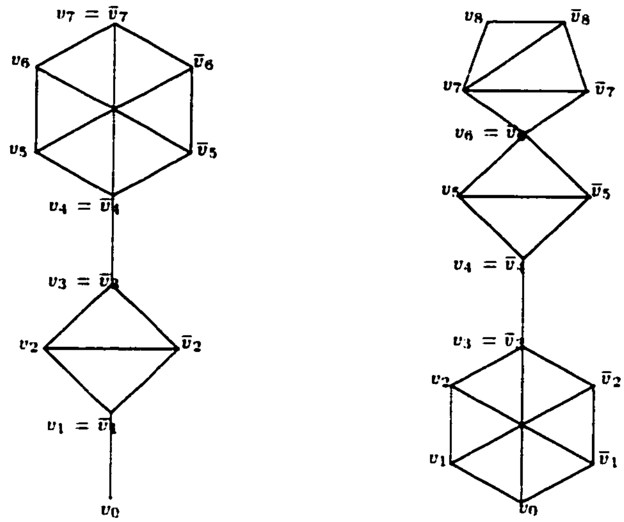

Figure 3.4: The figure on the left is a string of pearls of type I. The figure on the right is a string of pearls of type II.

and $\overline{v}_j, \overline{e}_j, \overline{v}_{j-1}, \dots, \overline{v}_{i+1}, \overline{e}_{i+1}, \overline{v}_i$ are combinatorial geodesics in D_{ij} from v_j to v_i. To see this, suppose that the combinatorial path $v_j, e_j, v_{j-1}, e_{j-1}, \dots, v_{i+1}, e_{i+1}, v_i$ of length $j - i$ is not a combinatorial geodesic from v_j to v_i. Then there is a combinatorial path $v_j, \tilde{e}_j, \tilde{v}_{j-1}, \dots, \tilde{v}_{j-r+1}, \tilde{e}_{j-r+1}, v_i$ from v_j to v_i of length $r < j - i$. Therefore $v_n, e_n, v_{n-1}, \dots, v_j, \tilde{e}_j, \tilde{v}_{j-1}, \dots, \tilde{v}_{j-r+1}, \tilde{e}_{j-r+1}, v_i, e_i, v_{i-1}, \dots, v_1, e_1, v_0$ is a combinatorial path from v_n to v_0 of length $n - (j - i) + r < n$ since $r < j - i$. This is a contradiction since $d_C(v_n, v_0) = n$. Thus $v_j, e_j, v_{j-1}, e_{j-1}, \dots, v_{i+1}, e_{i+1}, v_i$ is a combinatorial geodesic from v_j to v_i. Similarly, $\overline{v}_j, \overline{e}_j, \overline{v}_{j-1}, \overline{e}_{j-1}, \dots, \overline{v}_{i+1}, \overline{e}_{i+1}, \overline{v}_i$ is a combinatorial geodesic from $\overline{v}_j$ to $\overline{v}_i$. The second case we consider is when v_j and $\overline{v}_j$ are neighbors. Let e denote the edge spanned by v_j and $\overline{v}_j$. Then $v_j, e_j, v_{j-1}, \dots, v_{i+1}, e_{i+1}, v_i = \overline{v}_i, \overline{e}_{i+1}, \overline{v}_{i+1}, \dots, \overline{v}_{j-1}, \overline{e}_j, \overline{v}_j, e, v_j$ is a closed combinatorial path which bounds a disk D_{ij} in S. The same argument as before shows that $v_j, e_j, v_{j-1}, \ldots, v_{i+1}, e_{i+1}, v_i$ and $\overline{v}_j, \overline{e}_j, \overline{v}_{j-1}, \ldots, \overline{v}_{i+1}, \overline{e}_{i+1}, \overline{v}_i$ are combinatorial geodesics from v_j to v_i and from $\overline{v}_j$ to $\overline{v}_i$, respectively. In both cases, the interior vertices of D_{ij} are interior vertices of S, all of which have degree at least six. Therefore in the first case, D_{ij} is a geodesic disk of type I, and in the second case, D_{ij} is a geodesic disk of type II.

This brings us to the main result of the section.

Theorem 17. *If S is a string of pearls of type I then every vertex v of S lies on a combinatorial geodesic from v_n to v_0. If S is a string of pearls of type II then every vertex v of S lies on a combinatorial geodesic either from v_n to v_0 or from $\overline{v}_n$ to v_0.*

Proof. The proof is by induction on the number of 2-simplices F. If $F = 1$ then S must be a string of pearls of type II with $v_k = \overline{v}_k$ for $k = 0, \ldots, n-1$. These vertices together with v_n and $\overline{v}_n$ are the only vertices of S, and each of them is on a combinatorial geodesic by definition. If $F = 2$ then D must be a string of pearls of type I with $v_k \neq \overline{v}_k$ for exactly one $k \in \{0, 1, \ldots, n\}$. There are no interior vertices and each of the vertices is on a combinatorial geodesic by definition.

For general F, suppose first that S is a string of pearls of type I with at least three 2-simplices. Since $v_n = \overline{v}_n$ and $v_0 = \overline{v}_0$, there exist i and j with $i < j$, $v_i = \overline{v}_i$, $v_j = \overline{v}_j$, and $v_k \neq \overline{v}_k$ for $i < k < j$. Hence S must contain a subcomplex D_{ij} which is a geodesic disk of type I bounded by the combinatorial geodesics $v_j, e_j, v_{j-1}, e_{j-1}, \ldots, v_{i+1}, e_{i+1}, v_i$ and $\overline{v}_j, \overline{e}_j, \overline{v}_{j-1}, \overline{e}_{j-1}, \ldots, \overline{v}_{i+1}, \overline{e}_{i+1}, \overline{v}_i$ and such that the exterior vertices of D_{ij} are exterior vertices of S. By lemma 16, D_{ij} has an exterior vertex $v \notin \{v_i, v_j\}$ that satisfies deg $v = 3$, and this vertex is also an exterior vertex of S.

Second, suppose S is a string of pearls of type II with at least three 2-simplices. Let e denote the edge spanned by v_n and $\overline{v}_n$. Either S contains a subcomplex D_{ij} with at least three 2-simplices which is a geodesic disk of type II bounded by the edge e together with the combinatorial geodesics $v_n, e_n, v_{n-1}, \ldots, v_{i+1}, e_{i+1}, v_i$ and $\overline{v}_n, \overline{e}_n, \overline{v}_{n-1}, \ldots, \overline{v}_{i+1}, \overline{e}_{i+1}, \overline{v}_i$ where $v_i = \overline{v}_i$, or S has a subcomplex D_{ij} which is a geodesic disk of type I as in the previous case. If D_{ij} is type I then D_{ij}, and therefore S, has an exterior vertex v of degree 3 as described in the previous paragraph. If D_{ij} is type II, then lemma 16 implies D_{ij} has an exterior vertex $v \notin \{v_i, v_n, \overline{v}_n\}$ that satisfies deg $v = 3$, and this vertex is also an exterior vertex of S. Therefore, whether S is a string of pearls of type I or type II, S has an exterior vertex v of degree exactly three.

Without loss of generality assume $v = v_k$ for some $k \in \{1, \ldots, n-1\}$. Let v'_k denote the unique neighbor of v_k different from v_{k-1} and v_{k+1}. Let e'_k denote the edge spanned by v_{k-1} and v'_k, e'_{k+1} denote the edge spanned by v_{k+1} and v'_k, and e' denote the edge spanned by v_k and v'_k. Let α_1 denote the 2-simplex in S spanned by v_k, v'_k, and v_{k+1} and α_2 the 2-simplex spanned by v_k, v'_k, and v_{k-1}. Consider the subcomplex $S' = S - \{v_k, e', e_k, e_{k+1}, \alpha_1, \alpha_2\}$ of S. v'_k is a neighbor of both v_{k-1} and v_{k+1}, which are distances $k-1$ and $k+1$, respectively, from v_0. Thus it must be the case that $d(v'_k, v_0) = k$. Therefore the exterior vertices of S' are the vertices of the combinatorial geodesics $v_n, e_n, v_{n-1}, \ldots, v_{k+1}, e'_{k+1}, v'_k, e'_k, v_{k-1}, \ldots, v_1, e_1, v_0$ and $\overline{v}_n, \overline{e}_n, \overline{v}_{n-1}, \ldots, \overline{v}_1, \overline{e}_1, \overline{v}_0$ from v_n to v_0. The interior vertices of S' are interior vertices of S and thus all have degree at least 6. Therefore if S is a string of pearls of type I, then S' is again a string of pearls of type I and if S is a string of pearls of type II, then S' is again a string of pearls of type II. In either case, S' has two fewer 2-simplices than S. By induction, every interior vertex of S' lies on a geodesic from v_n to v_0. The only interior vertex of S not in S' is v'_k, which we have already seen lies on a combinatorial geodesic from v_n to v_0, and the proof is complete. □

In particular, theorem 17 applies to every geodesic disk. This brings us to the main result of this section. Every vertex on a combinatorial geodesic from v_n to v_0 must be closer to v_0 than v_n, so an immediate corollary of theorem 17 is that combinatorial distance in D, measured from v_0, is maximized on the boundary of D.

Corollary 18. *If S is a string of pearls of type either I or II, bounded by combinatorial geodesics $v_n, e_n, v_{n-1}, \ldots, v_1, e_1, v_0$ and $\overline{v}_n, \overline{e}_n, \overline{v}_{n-1}, \ldots, \overline{v}_1, \overline{e}_1, v_0$, then $d(v, v_0) < n$ for every interior vertex v of S.*

If v and v' are neighbors in S then $d(v, v_0)$ and $d(v', v_0)$ differ by at most one. In particular, corollary 18 implies that any neighbor v of v_n or $\overline{v}_n$ in the interior of a geodesic disk satisfies $d(v, v_0) = n - 1$.

Lemma 19. *Let J be a simplicial complex whose underlying space is homeomorphic to $\mathbb{R}^2$ such that $\deg v \geq 6$ for all $v \in J$. Let v^* be a distinguished vertex of J. Then if $d(v, v^*) = n$, v has at most two neighbors that are combinatorial distance $n - 1$ from v^*.*

Proof. The proof is by induction on $d(v, v^*)$. Suppose $d(v, v^*) = 1$. Then v^* is the only neighbor of v that is combinatorial distance 0 from v^*.

For the general case, suppose $d(v, v^*) = n$, and that v has more than two neighbors of combinatorial distance $n - 1$ from v^*. Let $u_1, \ldots, u_l$ be the neighbors of v that are distance $n - 1$ from v^* and consider a combinatorial geodesic $v, e, u_k, \ldots, v^*$ for each $k \in \{1, \ldots, l\}$. Together these combinatorial geodesics determine a subcomplex D of J which is a geodesic disk of type I and which contains all of $u_1, \ldots, u_l$. By theorem 17 and corollary 18, interior vertices of D are distance at most $n - 1$ from v^* and every interior vertex of D lies on a combinatorial geodesic from v to v^*. Therefore neighbors of v in J are combinatorial distance $n - 1$ from v^* if and only if they are in $\overline{\mathrm{St}}\ v\ \cap D$. Hence we may assume that $u_1, \ldots, u_l$ are labeled so that u_i and u_{i+1} are neighbors in $\overline{\mathrm{St}}\ v$ for $i = 1, \ldots, l - 1$. Denote the combinatorial geodesics bounding D by $v, e_n, v_{n-1}, e_{n-1}, \ldots, v_1, e_1, v^*$ and $v, \overline{e}_n, \overline{v}_{n-1}, \overline{e}_{n-1}, \ldots, \overline{v}_1, \overline{e}_1, v^*$, where $v_{n-1} = u_1$ and $\overline{v}_{n-1} = u_l$.

We will show that if u_i and u_j are neighbors in D then u_i and u_j are neighbors in $\overline{\mathrm{St}}\ v$. If u_i and u_j are neighbors in D, denote the edge they span by $e_{i,j}$. Assume $i < j$, and suppose that u_i and u_j are not neighbors in $\overline{\mathrm{St}}\ v$, so that $j - i \geq 2$. Consider the subcomplex A of D that is homeomorphic to a disk, bounded by the closed combinatorial path $u_i, e_{i,i+1}, u_{i+1}, \ldots, u_{j-1}, e_{j-1,j}, u_j, e_{j,i}, u_i$. Every exterior vertex of A except possibly u_i and u_j is an interior vertex of D and thus must have degree at least six in D. Since the only neighbor of u_k in $D - A$ is v for $i < k < j$, u_k must have degree at least five in A for $i < k < j$. In particular, $4 - \deg u_i < 0$ for exterior vertices u_k of A, $i < k < j$. The vertices u_i and u_j have degree at least two in A. Every interior vertex of A must have degree at least six since it is also an interior vertex of D. Equation 3.1 then implies

$$\begin{aligned}
6 &\leq \sum_{k=i}^{j}(4 - \deg u_k) \\
&= (4 - \deg u_i) + (4 - \deg u_j) + \sum_{k=i+1}^{j-1}(4 - \deg u_k) \\
&\leq 2 + 2 + \sum_{k=i+1}^{j-1}(4 - \deg u_k) \\
&= 4 + \sum_{k=i+1}^{j-1}(4 - \deg u_k) \\
&< 4
\end{aligned}$$

and we reach a contradiction. Thus if u_i and u_j are neighbors in D, the edge they span is in $\overline{\mathrm{St}}\ v$, as desired. This implies that u_k has at most two neighbors of combinatorial distance $n-1$ from v^*. For $i < k < j$, u_k is an interior vertex of D that has exactly three neighbors in $\overline{\mathrm{St}}\ v$, so u_k must have at least three neighbors in D that are not in $\overline{\mathrm{St}}\ v$. Thus u_k must have at least three neighbors that are combinatorial distance $n-2$ from v^*. This is a contradiction, since by induction u_k has at most two neighbors that are combinatorial distance $n-2$ from v^*. Therefore v has at most two neighbors of combinatorial distance $n-1$ from v^*. □

In particular, this implies that if D is a geodesic disk of type I bounded by combinatorial geodesics $v_n, e_n, v_{n-1}, e_{n-1}, \ldots, v_1, e_1, v_0$ and $v_n, \overline{e}_n, \overline{v}_{n-1}, \overline{e}_{n-1}, \ldots, \overline{v}_1, \overline{e}_1, v_0$,

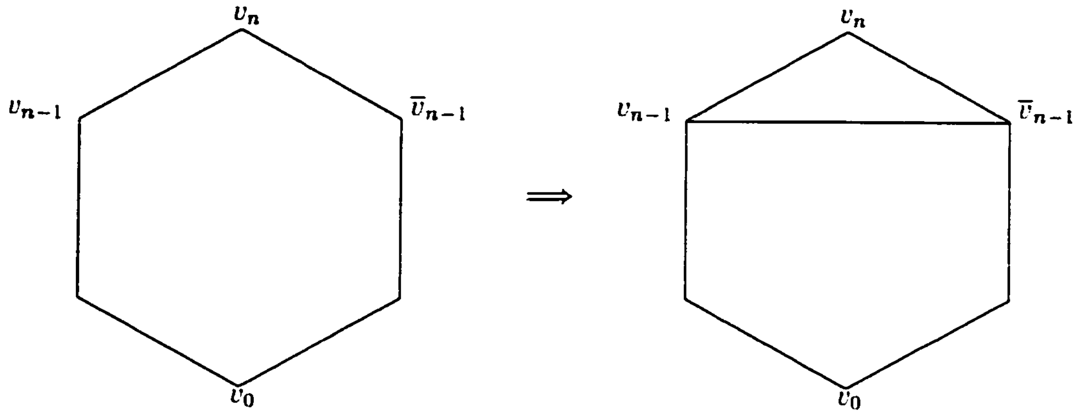

Figure 3.5: v_n, v_{n-1}, and $\overline{v}_{n-1}$ span a face in a geodesic disk of type I.

then any interior vertex v of D satisfies $d_C(v, v_0) \leq n-2$.

Corollary 20. *If D is a geodesic disk of type I, bounded by combinatorial geodesics $v_n, e_n, v_{n-1}, e_{n-1}, \ldots, v_1, e_1, v_0$ and $v_n, \overline{e}_n, \overline{v}_{n-1}, \overline{e}_{n-1}, \ldots, \overline{v}_1, \overline{e}_1, v_0$ from v_n to v_0, then v_n, v_{n-1}, and $\overline{v}_{n-1}$ span a 2-simplex in D.*

Proof. By the previous lemma, D has no interior vertices of combinatorial distance $n-1$ from v_0, implying v_n must have degree 2, which then implies v_{n-1} and $\overline{v}_{n-1}$ are neighbors. Let $\tilde{e}$ denote the edge spanned by v_{n-1} and $\overline{v}_{n-1}$. Then $v_n, e_n, v_{n-1}, \tilde{e}, \overline{v}_{n-1}, \overline{e}_n, v_n$ is a closed combinatorial path that bounds a disk in D, all of whose interior vertices must have degree at least six. By the combinatorial Gauss-Bonnet formula, $(4 - \deg v_n) + (4 - \deg v_{n-1}) + (4 - \deg \overline{v}_{n-1}) \geq 6$. Thus v_n, v_{n-1}, and $\overline{v}_{n-1}$ must all have degree 2, which implies they span a 2-simplex in D. $\square$

3.2 Collapsing a Triangulated Disk

Let D be any triangulated disk. We will now outline a proof that D simplicially collapses to a vertex by defining a discrete Morse function on D that has exactly one critical point. We will accomplish this by defining a discrete vector field V on D with no nontrivial closed V-paths. By theorem 13, V is then the gradient vector field of a discrete Morse function f. We will verify that f has exactly one critical point, and hence, by theorem 7, D simplicially collapses to a point.

Let v^* be a distinguished vertex of D. Define $V(v^*) = 0$. For $v \neq v^*$ define $V(v) = e$ where $v, e, \ldots, v^*$ is any combinatorial geodesic from v to v^*. Now if an edge e is in the image of V, define $V(e) = 0$. If e is not in the image of V then consider the string of pearls bounded by e and the combinatorial geodesics from the endpoints of e to v^*. Define $V(e) = \sigma$ where σ is the unique 2-simplex in the interior of the geodesic disk and containing e as a face.

To see that V is a discrete vector field we need to verify that for any edge e and σ, if $V(e) = \sigma$ then there is no other edge e' with $V(e') = \sigma$. If the geodesic disk bounded by e and the combinatorial geodesics from the endpoints of e to v^* is a geodesic disk of type I then one endpoint v_n of e is combinatorial distance n from v^* and the other endpoint v_{n-1} is combinatorial distance $n-1$ from v^*. Lemma 19 implies that $V(v_n)$ is an edge which is a face of σ, and therefore this edge does not map to σ. By lemma 20 the edge in σ different from e and $V(v_n)$ has endpoints that are both distance $n-1$ from v^* and opposite vertex v_n in σ. By lemma 18, this edge can not map to σ. Therefore e is the unique edge mapping to σ. If the geodesic disk bounded by e and the combinatorial geodesics from the endpoints of e to v^* is a geodesic disk of type II then by lemma 18, neither of the edges in σ different from e can map to σ since each of these two edges has an endpoint combinatorial distance $n-1$ from v^* and an opposite vertex in σ combinatorial distance n from v^*. Again, e is the unique edge mapping to σ. Therefore V is a discrete vector field on D. Furthermore, there are no nontrivial closed V-paths since arrows of the vector field point to vertices closer to v^*. (A formal proof of this fact will be given in section 3.) Theorem 13 implies that V is the gradient vector field of a discrete Morse function. Recall that the critical simplices of the discrete Morse function corresponding to a discrete vector field on K are precisely the simplices $\alpha \in K$ such that α is not in the image of U and $U(\alpha) = 0$. In figure 3.6 we indicate that $V(\alpha) = \beta$ with an arrow pointing from α to β. By definition then, the critical simplices of D are the simplices that are neither the head nor the tail or an arrow. Thus v^* is the only critical vertex of D, and D has no critical edges. In section 2.2 we defined m_p to be the number of critical simplices of dimension p. Referring to the weak Morse inequalities on page 14 we see that

$$1 = \chi(D) = m_0 - m_1 + m_2 = 1 - 0 + m_2,$$

which implies that $m_2 = 0$ and therefore D has no critical 2-simplices. Thus the discrete Morse function has exactly one critical point. By theorem 7, D simplicially collapses to v^*.

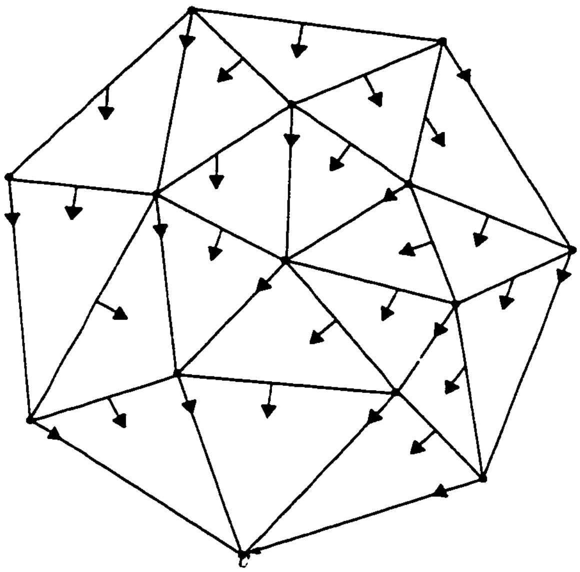

Figure 3.6: The gradient vector field corresponding to a discrete Morse function on D.

3.3 The Geometry of a 3-Dimensional CAT(0) Simplicial Complex

The method described in section 3.2 for defining a discrete Morse function will be our guide in creating a Morse function on a three-dimensional complex. Motivated by the need to find a clear direction in which to flow from a noncritical simplex, we again create a Morse function that is also a distance function in a combinatorial sense. In order to find a good direction in which to flow from an edge, we consider knots in K formed by combinatorial geodesics. Because CAT(0) implies that K is simply connected, for each knot there exists a simplicial map of a triangulated disk into K which maps the boundary of the disk to the knot (see section 2.1. The pivotal step is showing that any such disk, which is minimal with respect to the number of triangles, must also be CAT(0). This is a combinatorial analog of the fact that a minimal surface in $\mathbb{R}^3$ must be nonpositively curved. For each edge, the associated minimal disk provides a direction in which to flow. The problem is thus reduced to two dimensional CAT(0) disks, which we understand.

We begin with some elementary Euclidean geometry that will be used to compare geodesic triangles in the CAT(0) space to comparison triangles in $\mathbb{R}^2$ in the proofs to follow.

Lemma 21. *Consider two Euclidean triangles. The first triangle has side lengths A, B, and C, and the second triangle has side lengths A', B, and C. Let α and α' denote the angles opposite the sides of lengths A and A', respectively. Then $A' > A$ if and only if $\alpha' > \alpha$.*

Proof. By the law of cosines, $A^2 = B^2 + C^2 - 2BC\cos\alpha$. Differentiating with respect to A yields $2A = 2BC\sin\alpha\frac{d\alpha}{dA}$, or $\frac{d\alpha}{dA} = \frac{A}{BC\sin\alpha} > 0$ since $0 < \alpha < \pi$. Thus α increases if and only if A increases. Equivalently, $A' > A$ if and only if $\alpha' > \alpha$. □

Corollary 22. *Consider two Euclidean triangles, $\Delta_1 = (a, b, c)$ and $\Delta_2 = (\overline{a}, \overline{b}, \overline{c})$. Suppose A, B, and C are the lengths of the sides in Δ_1 opposite a, b, and c, respectively, and A, D, and D are the lengths of the sides in Δ_2 opposite $\overline{a}, \overline{b}$, and $\overline{c}$, respectively. Let α denote the angle at a and $\overline{\alpha}$ the angle at $\overline{a}$. If $B > D$ and $C > D$ then $\alpha < \overline{\alpha}$.*

Proof. Without loss of generality, assume $C > B$. Let p_1 and q be the points on the edge between a and b satisfying $d(a, p_1) = D$ and $d(a, q) = B$, and let p_2 be the point on the edge between a and c satisfying $d(a, p_2) = D$. Then triangle (a, c, q) is isosceles and $d(p_1, p_2) < d(c, q)$. Let β denote the angle at q in triangle (a, c, q), γ the angle at q in triangle (b, c, q), and δ the angle at b in triangle (a, b, c). Since triangle (a, c, q) is isosceles we have $\beta = \frac{\pi - \alpha}{2}$ and thus $\gamma = \frac{\pi + \alpha}{2}$. Then δ must be less than γ; otherwise the sum of the interior angles of triangle (b, c, q) would be greater than π. Then $\delta < \gamma$ implies $d(c, q) < A$. Thus $d(p_1, p_2) < d(c, q) < A$, and the lemma applied to triangles (a, p_1, p_2) and Δ_2 implies $\alpha < \overline{\alpha}$. □

Lemma 23. *Let K be a 3-dimensional simplicial complex endowed with the piecewise Euclidean geometry given by declaring edges to have unit length. Then each of the following holds:*

1. *If w is a vertex of K then $B_{\sqrt{\frac{2}{3}}}(w) \subseteq \overline{St}\ w$.*

2. *Let w be a vertex of K and p the midpoint of an edge in K. Then $d(p, w) \leq \frac{\sqrt{3}}{2}$ if and only if $p \in \overline{St}\ w$.*

3. *Let w and w' be distinct vertices of K. If $d(w, w') < \sqrt{\frac{8}{3}}$ then $d(w, w') = 1$.*

Proof. 1. The distances from a vertex to the barycenter of its opposite face in a 1-, 2-, and 3-simplex are, respectively, 1, $\frac{\sqrt{3}}{2}$, and $\sqrt{\frac{2}{3}}$. Therefore in a three-dimensional simplicial complex, the distance from a vertex to the boundary of its star is at least $\sqrt{\frac{2}{3}}$, and hence $B_{\sqrt{\frac{2}{3}}}(w) \subseteq \overline{\mathrm{St}}\ w$.

2. To prove the forward direction, suppose that $p \notin \overline{\mathrm{St}}\ w$. Let w' be a vertex of K such that $p \in \mathrm{St}\ w'$. Then $\mathrm{St}\ w \cap \mathrm{St}\ w' = \emptyset$ and $d(p, \mathrm{Bd}(\mathrm{St}\ w')) = \frac{1}{\sqrt{6}}$. Also,

by part 1 of the proof, $d(w, \mathrm{Bd}(\overline{\mathrm{St}}\ w)) = \sqrt{\frac{2}{3}}$. The geodesic from p to w must leave the star of w' and enter the star of w, so $d(p, w) \geq \frac{1}{\sqrt{6}} + \sqrt{\frac{2}{3}} > \frac{\sqrt{3}}{2}$, which is a contradiction. Therefore $p \in \overline{\mathrm{St}}\ w$. Now we prove the other direction. If $p \in \overline{\mathrm{St}}\ w$ then either $p \in \mathrm{St}\ w$ or $p \in \mathrm{Lk}\ w$. If $p \in \mathrm{St}\ w$ then $d(p, w) = \frac{1}{2}$. If $p \in \mathrm{Lk}\ w$ then $d(p, w) = \frac{\sqrt{3}}{2}$.

3. By part 1 of the proof, $d(w, \mathrm{Bd}(\overline{\mathrm{St}}\ w)) = d(w', \mathrm{Bd}(\overline{\mathrm{St}}\ w')) = \sqrt{\frac{2}{3}}$. If $w' \notin \overline{\mathrm{St}}\ w$ then $\mathrm{St}\ w \cap \mathrm{St}\ w' = \emptyset$, so the geodesic from w to w' must leave the star of w' and enter the star of w. Thus $d(w, w') \geq 2\sqrt{\frac{2}{3}} = \sqrt{\frac{8}{3}}$, a contradiction. Thus $w' \in \overline{\mathrm{St}}\ w$, which implies $d(w, w') = 1$.

□

The following lemma begins our investigation of immersed CAT(0) disks in a 3-dimensional CAT(0) complex.

Lemma 24. *Suppose K is a simplicial complex whose underlying space $|K|$ is simply connected and $w_1, e_1, \ldots, w_k, e_k, w_{k+1} = w_1$ is a closed combinatorial path in K. There exists a triangulated k-gon D with exterior vertices $v_1, \ldots, v_k, v_{k+1} = v_1$, labeled so that v_{i+1} is a neighbor of v_i for $i = 1, \ldots, k$, and a simplicial map $|\phi| : |D| \to |K|$ satisfying $\phi(v_i) = w_i$ for $i = 1, \ldots, k$.*

Proof. Let $A \subseteq \mathbb{R}^2$ be any triangulated k-gon with distinct exterior vertices $v_1, \ldots, v_k, v_{k+1} = v_1$, labeled so that v_{i+1} is a neighbor of v_i for $i = 1, \ldots, k$. Let $\psi : \mathrm{Bd}A^0 \to K^0$ be the vertex map $\psi(v_i) = w_i$ for $i = 1, \ldots, k$. Recall the definition of a combinatorial path which says that two consecutive vertices in a combinatorial

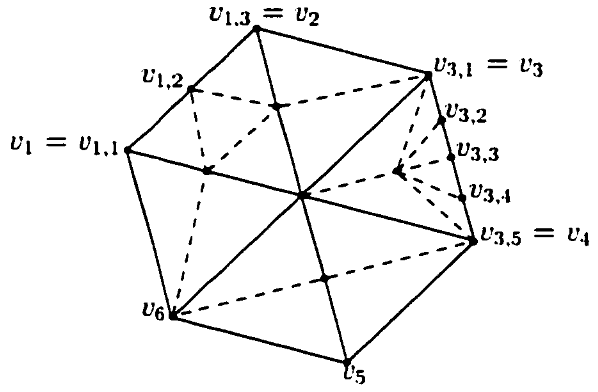

Figure 3.7: An example for $k = 6$. The dashed lines indicate the subdivision A'.

path must be distinct. Therefore the induced simplicial map $|\psi| : \mathrm{Bd}|A| \to |K|$ maps exterior edges of A to edges of K. Because $|K|$ is simply connected, $|\psi|$ extends to a continuous map $f : |A| \to |K|$ [Mul]. Consider the open covering $\mathcal{U} = \{f^{-1}(\mathrm{St}\ w) \mid w \in K\}$ of A. By results from Spanier, there exists a subdivision A' of A that is finer than $\mathcal{U}$. Then for every vertex $a \in A'$, $\mathrm{St}\ a \subset f^{-1}(\mathrm{St}\ w)$ for some $w \in K$, so the vertex map given by $\phi'(a) = w$ induces a simplicial map $|\phi'| : |A'| \to |K|$. Since for each $a \in A'$, $f(\mathrm{St}\ a) \subset \mathrm{St}\ w$ for some $w \in K$, $|\phi'|$ is a simplicial approximation to f.

Let c_i denote the 1-simplex spanned by the exterior vertices v_i and v_{i+1} of A. If c_i is not a simplex of A', then let $v_i = v_{i,1}, \ldots, v_{i,m} = v_{i+1}$ denote the vertices of A' that are points of c_i, labeled as in figure 3.3, so that $v_{i,j}$ is a neighbor of $v_{i,j+1}, j = 1, \ldots, m-1$. Since $f^{-1}(\mathrm{St}\ w_i)$ and $f^{-1}(\mathrm{St}\ w_{i+1})$ are the only sets in $\mathcal{U}$ that contain $v_{i,j}$ for $j = 2, \ldots, m-1$, we may assume that the subdivision A' is fine enough so that $\mathrm{St}\ v_{i,j} \subset f^{-1}(\mathrm{St}\ w_i)$ and thus assume that $\phi'(v_{i,j}) = w_i$, $j = 2, \ldots, m-1$.

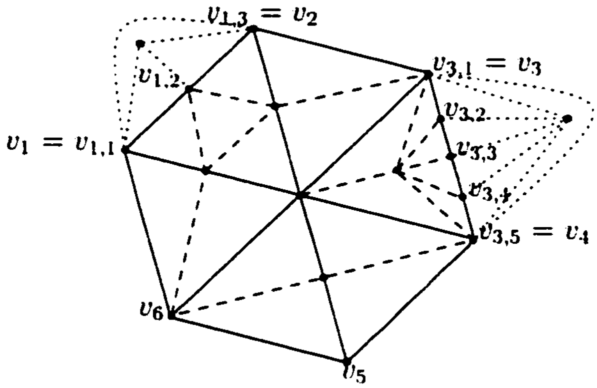

Figure 3.8: The k-gon D

Now we define a simplicial complex D that contains A' as a subcomplex and has the desired properties. The vertices of D are the vertices of A' together with the vertices $\overline{v}_i$ for each i such that c_i is not a simplex of A'. The 2-simplices of D are the 2-simplices of A' together with the simplices spanned by the sets of vertices $\{\overline{v}_i, v_i, v_{i+1}\}$ and $\{\overline{v}_i, v_{i,j}, v_{i,j+1}\}$ for each i such that c_i is not a simplex of A' and $j = 1, \ldots, m-1$.

Now the exterior vertices of D are $v_1, \ldots, v_k, v_{k+1} = v_1$ and v_{i+1} is a neighbor of v_i for $i = 1, \ldots, k$. We extend the map ψ' to a map $\phi : D^0 \to K^0$ by defining $\phi(\overline{v}_i) = w_i$. Then each 2-simplex of D that is not a 2-simplex of A' is spanned by three vertices all of which map to at most two (neighboring) vertices in K. Therefore $|\phi| : |D| \to |K|$ is a simplicial map, as desired. $\square$

In the next lemma we prove some properties of any triangulated disk as described in lemma 24, which is minimal in the sense that it has the smallest possible number of 2-simplices. Such a disk exists since K has a finite number of simplices. For the lemma we make a new definition. Say that two 2-simplices β_1 and β_2 are *neighbors* if there exists an edge e with $e < \beta_1$ and $e < \beta_2$.

Lemma 25. *If D is a simplicial complex as described in lemma 24, which is minimal with respect to the number of 2-simplices, then*

1. *$|\phi| : |D| \to |K|$ maps 0-, 1-, and 2-simplices to 0-, 1-, and 2-simplices, respectively.*

2. *$\beta_1^{(2)}$ and $\beta_2^{(2)}$ are neighbors in D implies $|\phi|(\beta_1) \neq |\phi|(\beta_2)$.*

Proof. 1. Suppose that $|\phi|$ maps a 2-simplex to an edge, a 2-simplex to a vertex, or an edge to a vertex. In each case, $|\phi|$ maps an interior edge to a vertex. We will show that this can not occur. Suppose that vertices a and b are neighbors and that $\phi(a) = \phi(b)$. Then the edge spanned by a and b must be an interior edge. Let c and d be the vertices of D such that a, b, and c span a 2-simplex and a, b, and d span a 2-simplex. Let U be the set of vertices $\{a, b\} \cup \{u \mid u \text{ is a neighbor of both } a \text{ and } b\}$ in D and let L_1 be the subcomplex of D consisting of all simplices spanned by vertices from the set U. Define L_2 to be the subcomplex of D consisting of all simplices in $D - L_1$ that do not contain b as a face. Finally, define L_3 to be the set of 2-simplices spanned by the set $\{u, v, a \mid u, v, \text{and } b \text{ span a 2-simplex in } D - L_1\}$ along with all of their faces. Then $(D - L_1) \cup L_2 \cup L_3$ gives a simplicial complex that also satisfies the properties of the previous lemma but that has at least two fewer triangles than D, a contradiction. Therefore $|\phi|$ maps simplices of dimension p to simplices of dimension p.

2. Suppose that a, b, c, and d are vertices in D such that a, b, and c span a 2-simplex β_1 and b, c, and d span a 2-simplex β_2. Then $|\phi|(\beta_1) = |\phi|(\beta_2)$ if and only if $\phi(a) = \phi(d)$. Suppose $\phi(a) = \phi(d)$. Let U be the set of vertices $\{a, d\} \cup \{u \mid u \text{ is a neighbor of both } a \text{ and } d\}$ and let L_1 be the subcomplex of D consisting of simplices spanned by vertices from the set U. Define L_2 to be the subcomplex of D consisting of all simplices in $D - L_1$ that do not contain d as a face. Finally, define L_3 to be the set of 2-simplices spanned by the set $\{u, v, a \mid u, v, \text{and } d \text{ span a 2-simplex in } D - L_1\}$ along with all of their faces. Then $(D - L_1) \cup L_2 \cup L_3$ gives a simplicial complex that also satisfies the properties of the previous lemma but that has at least two fewer triangles than D, a contradiction. Therefore $|\phi|(\beta_1) \neq |\phi|(\beta_2)$.

□

Suppose that D is a triangulated disk that maps into K as described in lemma 24 and which is minimal with respect to the number of 2-simplices. Endow D with the piecewise Euclidean geometry given by declaring edges to have unit length. Our current goal is to prove that D must be a CAT(0) disk. We will do this by showing that D has no interior vertices of degree 3, 4, or 5. The first step is to show that any three neighbors in K must span a 2-simplex.

Lemma 26. *If w_1, w_2, and w_3 are vertices in K with w_i a neighbor of w_j for $i \neq j$, then w_1, w_2, and w_3 span a 2-simplex in K.*

Proof. Let e be the edge spanned by w_2 and w_3, and m the unique point on e such that $d(w_2, m) = d(w_3, m)$. If $m \in \overline{\mathrm{St}}\ w_1$ then $e \in \overline{\mathrm{St}}\ w_1$ and therefore w_1, w_2, and w_3 span a 2-simplex in $\overline{\mathrm{St}}\ w_1$. To see that $m \in \overline{\mathrm{St}}\ w_1$, let $\Delta' = (w_1', w_2', w_3')$ be a comparison triangle for Δ, and m' the point in Δ' satisfying $d_{\mathbb{R}^2}(w_2', m') = d_{\mathbb{R}^2}(w_3', m')$. The CAT(0) inequality implies $d(w_1, m) \leq d_{\mathbb{R}^2}(w_1', m') = \frac{\sqrt{3}}{2}$. Now m is the midpoint of an edge and contained in $B_{\frac{\sqrt{3}}{2}}(w_1)$. By lemma 23, $m \in \overline{\mathrm{St}}\ w_1$, as desired. $\square$

We are now ready to prove that $|D|$ is CAT(0).

Theorem 27. *Let D be a triangulated disk as in lemma 24 that is minimal with respect to the number of 2-simplices. If D has the piecewise Euclidean geometry endowed by declaring edges to have unit length then $|D|$ is a CAT(0) space.*

Proof. It suffices to show that D has no interior vertices of degree 3, 4, or 5.

Suppose v is an interior vertex of D with $\deg v = 3$. Let v_1, v_2, and v_3 be the three distinct neighbors of v, and σ_1, σ_2, and σ_3 the three distinct 2-simplices in D that contain v as a face. By lemma 25, $|\phi| : |D| \to |K|$ preserves the dimension of each simplex, so the vertices $\phi(v_1), \phi(v_2)$, and $\phi(v_3)$ are distinct, neighboring vertices in K, and thus, by lemma 26 span a 2-simplex β in K. Let L_1 be the subcomplex of D that consists of the 2-simplices σ_1, σ_2, and σ_3 and all their faces. Let L_2 be the subcomplex of D consisting of the simplices of $D - L_1$ and all their faces. Let α be a 2-simplex spanned by the vertices v_1, v_2, and v_3. Then $D' = L_2 \cup \{\alpha\}$ is a simplicial complex whose underlying space is a disk and the vertex map ϕ restricted to the vertices of D' induces a simplicial map from $|D'|$ to $|K|$ satisfying the hypotheses of corollary 24. But D' has two fewer 2-simplices than D, a contradiction since D is minimal with respect to the number of 2-simplices. Therefore D has no vertices of degree 3.

Suppose v is an interior vertex of D with deg v=4. Let v_1, v_2, v_3, and v_4 be the four distinct neighbors of v, with v_i a neighbor of v_{i+1}, $i = 1, 2, 3$. Let $\sigma_1, \sigma_2, \sigma_3$, and σ_4 be the four distinct 2-simplices in D that contain v as a face. Denote the image of $\phi(v_i)$ by w_i for $i = 1, 2, 3, 4$. By lemma 25, the vertices w_1, w_2, w_3, and w_4 are all distinct vertices in K. We will show that either $d(w_1, w_3) < \sqrt{2}$ or $d(w_2, w_4) < \sqrt{2}$. Since $\sqrt{2} < \sqrt{\frac{8}{3}}$, lemma 23, part 3 then implies that one of these distances is equal to 1. Let $2A = d(w_1, w_3)$. Geodesics between two points in a CAT(0) space are unique so let s be the unique point in $|K|$ satisfying $d(w_1, s) = d(w_3, s) = A$. Let $d(w_2, s) = B$ and $d(w_4, s) = C$. Let $\Delta' = (w_1', w_2', w_3')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\Delta = (w_1, w_2, w_3)$ in $|K|$, s' the point on the edge spanned by w_1' and w_3' satisfying $d_{\mathbb{R}^2}(w_1', s') = d_{\mathbb{R}^2}(w_3', s') = A$, and D the distance from w_2' to s'. Δ' is isosceles, so $A^2 + D^2 = 1$. If $A \leq \frac{1}{\sqrt{2}}$ then $d(w_1, w_3) = 2A \leq \sqrt{2}$. If $A \geq \frac{1}{\sqrt{2}}$ then since $|K|$ is CAT(0) we have $B \leq D = \sqrt{1 - A^2} \leq \sqrt{1 - (\frac{1}{\sqrt{2}})^2} = \frac{1}{\sqrt{2}}$ and similarly, $C \leq \frac{1}{\sqrt{2}}$. Hence $d(w_2, w_4) \leq \sqrt{2}$. So either $d(w_1, w_3) \leq \sqrt{2}$ or $d(w_2, w_4) \leq \sqrt{2}$. We will assume without loss of generality that $d(w_1, w_3) \leq \sqrt{2}$. By lemma 23, $d(w_1, w_3) = 1$. Lemma 26 then implies that w_1, w_2, and w_3 span a 2-simplex β_1 and w_1, w_3, and w_4 span a 2-simplex β_2 in K. Let L_1 be the subcomplex of D that consists of the 2-simplices $\sigma_1, \sigma_2, \sigma_3$, and σ_4 and all their faces. Let L_2 be the subcomplex of D consisting of the simplices of $D - L_1$ and all their faces. Let α_1 be a 2-simplex spanned by the vertices v_1, v_2, and v_3 and let α_2 be a 2-simplex spanned by the vertices v_1, v_3, and v_4. Then $D' = L_2 \cup \{\alpha_1, \alpha_2\}$ is a simplicial complex whose underlying space is a disk satisfying the conditions of corollary 24, and with two fewer 2-simplices than D, a contradiction since D is minimal with respect to the number of 2-simplices. Therefore D has no vertices of degree 4.

Suppose v is an interior vertex of D with deg v=5. Let v_1, v_2, v_3, v_4, and v_5 be the five distinct neighbors of v, with v_i a neighbor of v_{i+1} for $i = 1, 2, 3, 4$. Again by lemma 25, the vertices w_1, w_2, w_3, w_4, and w_5 are all distinct vertices in K. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, and σ_5 be the five distinct 2-simplices in D that contain u as a face. Denote the image of $\phi(v_i)$ by w_i for $i = 1, 2, 3, 4, 5$. Let $L = 2\sin(\frac{3\pi}{10})$. Again using lemma 23, we will show that $d(w_i, w_j) = 1$ in K for some non-neighboring vertices w_i and w_j by showing that $d(w_i, w_j) < L < \sqrt{\frac{8}{3}}$. Lemma 23 then implies $d(w_i, w_j) = 1$. Without loss of generality, we will show that if both $d(w_1, w_3) > L$ and $d(w_3, w_5) > L$, then $d(w_2, w_4) < L$.

Let a be the point on the geodesic segment from w_1 to w_3 that satisfies $d(w_3, a) = L - 1$ and b the point on the geodesic segment from w_3 to w_5 that satisfies $d(w_3, b) = L - 1$. Consider the geodesic triangle $\Delta = (w_1, w_2, w_3)$ in $|K|$, and let $\Delta' = (w_1', w_2', w_3')$ be a comparison triangle for Δ in $\mathbb{R}^2$. Let α' be the angle at vertex w_2' and γ' the angle at w_1' and w_3'. Let a' be the point on Δ' satisfying $d_{\mathbb{R}^2}(w_3', a') = L - 1$. By the CAT(0) inequality, $d(w_2, a) \leq d(w_2', a')$.

Consider a Euclidean triangle Δ'' with vertices w_1'', w_2'', and w_3'' satisfying $d(w_1'', w_2'') = d(w_2'', w_3'') = 1$ and $d(w_1'', w_3'') = L$. Let α'' be the angle at w_2'' and γ'' the angle at w_1'' and w_3''. Let a'' be the point on the edge between w_1'' and w_3'' satisfying $d(w_3'', a'') = L - 1$. Then $d(w_2'', a'') = L - 1$. By lemma 21, $d(w_1', w_3') > d(w_1'', w_3'')$ implies $\alpha' > \alpha''$. Therefore $\gamma' < \gamma''$, which implies, again by lemma 21, that $d(w_2', a') < d(w_2'', a'')$. Therefore we have $d(w_2, a) \leq d(w_2', a') < d(w_2'', a'') = L - 1$.

Similarly we can show that $d(w_4, b) < L - 1$.

Now we consider $d(a, b)$. Consider the geodesic triangle $\overline{\triangle} = (w_1, w_3, w_5)$ in K, and let $\overline{\triangle}' = (\overline{w}_1', \overline{w}_3', \overline{w}_5')$ be a comparison triangle for $\overline{\triangle}$ in $\mathbb{R}^2$. Let β' be the angle at vertex $\overline{w}_3'$. Let a' be the point on the edge spanned by $\overline{w}_1'$ and $\overline{w}_3'$ satisfying $d_{\mathbb{R}^2}(\overline{w}_3', a') = L - 1$ and b' the point on the edge spanned by $\overline{w}_5'$ and $\overline{w}_3'$ satisfying $d_{\mathbb{R}^2}(\overline{w}_3', b') = L - 1$. By the CAT(0) inequality, $d(a, b) \leq d(a', b')$.

Now consider a Euclidean triangle $\overline{\triangle}''$ with vertices $\overline{w}_1'', \overline{w}_3''$, and $\overline{w}_5''$ satisfying $d(\overline{w}_1'', \overline{w}_3'') = d(\overline{w}_3'', \overline{w}_5'') = L$ and $d(\overline{w}_1'', \overline{w}_5'') = 1$. Let β'' be the angle at $\overline{w}_3''$. Let a'' be the point on the edge spanned by $\overline{w}_1''$ and $\overline{w}_3''$ such that $d(\overline{w}_3'', a'') = L - 1$ and let b'' be the point on the edge spanned by $\overline{w}_3''$ and $\overline{w}_5''$ such that $d(\overline{w}_3'', b'') = L - 1$. Then $d(a'', b'') = 2 - L$. By corollary 22, $\beta' < \beta''$. Lemma 21 then implies that $d(a', b') < d(a'', b'')$. Therefore $d(a, b) \leq d(a', b') < d(a'', b'') = 2 - L$.

Now the geodesic segments in $|K|$ from w_2 to a, a to b, and b to w_4 give a path from w_2 to w_4. Thus we have $d(w_2, w_4) \leq d(w_2, a) + d(a, b) + d(b, w_4) < (L - 1) + (2 - L) + (L - 1) = L = 2\sin(\frac{3\pi}{10}) < \sqrt{\frac{8}{3}}$. By lemma 23 $d(w_2, w_4) = 1$, which implies that w_2, w_3, and w_4 span a 2-simplex β_1 in K. By the preceding argument for vertices of degree 4, either $d(w_1, w_4) = 1$ or $d(w_2, w_5) = 1$. Without loss of generality we assume that $d(w_1, w_4) = 1$, so w_1, w_2, and w_4 span a 2-simplex β_2 and w_1, w_4, and w_5 span a

2-simplex β_3 in K.

Let L_1 be the subcomplex of D that consists of the 2-simplices $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, and σ_5 and all their faces. Let L_2 be the subcomplex of D consisting of the simplices of $D - L_1$ and all their faces. Let α_1 be a 2-simplex spanned by the vertices v_2, v_3, and v_4, α_2 a 2-simplex spanned by the vertices v_1, v_2, and v_4, and α_3 a 2-simplex spanned by the vertices v_1, v_4, and v_5. Then $D' = L_2 \cup \{\alpha_1, \alpha_2, \alpha_3\}$ is a simplicial complex whose underlying space is a disk satisfying the conditions of corollary 24 with two fewer 2-simplices than D, a contradiction since D is minimal with respect to the number of 2-simplices. Therefore D has no vertices of degree 5. $\square$

Lemma 28. *Suppose that $w_n, e_n, w_{n-1}, \ldots, w_1, e_1, w_0$ and $\overline{w}_m, \overline{e}_m, \overline{w}_{m-1}, \ldots, \overline{w}_1, \overline{e}_1, w_0$ are combinatorial geodesics in K, and w_n and $\overline{w}_m$ are neighbors. Let D be any simplicial disk mapping to K via the map $|\phi|$ as in lemma 24 that is minimal with respect to the number of 2-simplices. Suppose v_n and $\overline{v}_m$ are the two exterior vertices of D mapping to w_n and $\overline{w}_m$, respectively. Let β denote the unique 2-simplex of D containing the vertices v_n and $\overline{v}_m$ and let w be the vertex of $|\phi|(\beta)$ different from w_n and $\overline{w}_m$. Then $d_C(w, w_0) = n - 1$.*

Proof. w is a neighbor of w_n so $d(w, w^*) \geq n - 1$. Let v be the vertex of β different from v_n and $\overline{v}_m$. We have seen that $d(v, v_0) = n - 1$. Let $v, \tilde{e}_{n-1}, \tilde{v}_{n-2}, \ldots, \tilde{v}_1, \tilde{e}_1, v_0$ be any combinatorial geodesic in D from v to v_0. Since the simplicial map $|\phi|$ preserves the dimension of each simplex, $|\phi|(v), |\phi|(\tilde{e}_{n-1}), |\phi|(\tilde{v}_{n-2}), \ldots, |\phi|(\tilde{v}_1), |\phi|(\tilde{e}_1), |\phi|(v_0)$ is a combinatorial path of length $n-1$ from w to w^* in K, which implies $d(w, w^*) \leq n-1$.

Therefore $d(w, w^*) = n - 1$. □

Lemma 28 shows us that at least some of the distance function information carries over from D to K. This will be exactly what we need in the next section to prove the main result of the chapter.

3.4 Proof of the Main Theorem

In this section we give a proof of the main theorem. stated on page 49. and written here again for easy reference.

Theorem 29. *Let K be a finite 3-dimensional simplicial complex endowed with the piecewise Euclidean geometry given by declaring edges to have unit length. and satisfying the additional property that every 2-simplex of K is a face of at most two 3-simplices of K. If $|K|$ is CAT(0) then K simplicially collapses to a point.*

We begin as we did in section 3.2. when showing that every triangulated disk simplicially collapses to a point. We first define a discrete vector field W on K such that every edge and all but one vertex either maps to another simplex of K or is in the image of W. We then show that there are no nontrivial closed W-paths. and hence, by theorem 13, the discrete vector field is the gradient vector field of a discrete Morse function f. By the definition of W. f has no critical edges and exactly one critical vertex.

In the two-dimensional case, we were able to immediately conclude that there were no critical 2-simplices and therefore, the disk collapsed to a point. In the three-dimensional case we are left with critical 2- and 3-dimensional simplices that we must still deal with. A similar Euler characteristic argument indicates that the number of critical 2-simplices is equal to the number of critical 3-simplices. We then show how to use theorem 14 to "cancel" out these critical simplices in pairs.

As before, we start by defining a function $W : K \to K \cup \{0\}$ by arbitrarily choosing one vertex of the complex to be a distinguished vertex. We denote this vertex by w^*. W is defined on vertices as follows. For each vertex $w \neq w^*$, choose any e such that $w, e, \dots, w^*$ is a combinatorial geodesic from w to w^*, (such a path exists because $|K|$ is connected) and let $W(w) = e$. Define $W(w^*) = 0$.

Now we define W on edges as follows. Let e be an edge in K. If there exists $w \in K$ with $W(w) = e$ then define $W(e) = 0$. If there does not exist $w \in K$ with $W(w) = e$ then let w_n and $\overline{w}_m$ denote the endpoints of e, and let $w_n, e_n, w_{n-1}, \dots, w_1, e_1, w^*$ and $\overline{w}_m, \overline{e}_m, \overline{w}_{m-1}, \dots, \overline{w}_1, \overline{e}_1, w^*$ be the combinatorial paths from w_n to w^* and $\overline{w}_m$ to w^*, respectively, satisfying $W(w_i) = e_i$ and $W(\overline{w}_i) = \overline{e}_i$ for all i. By lemma 24, there exists a minimal (in the sense of lemma 25) triangulated disk D with $n + m + 1$ distinct exterior vertices $v^*, v_1, \dots, v_n, \overline{v}_1, \dots, \overline{v}_m$, and a simplicial map $|\phi| : D \to K$ that satisfies $\phi(v^*) = w^*, \phi(v_i) = w_i$, and $\phi(\overline{v}_i) = \overline{w}_i$ for all i. There is a unique 2-simplex β in D such that $v_n < \beta$ and $\overline{v}_m < \beta$ and by lemma 25, $|\phi|(\beta)$ is a 2-simplex of K that contains the edge e as a face. Define $W(e) = |\phi|(\beta)$. Define $W(\alpha) = 0$ for all simplices α of dimension greater than or equal to two.

Theorem 30. *The function $W : K \to K \cup \{0\}$ is the gradient vector field of a Morse function with exactly one critical vertex and no critical edges.*

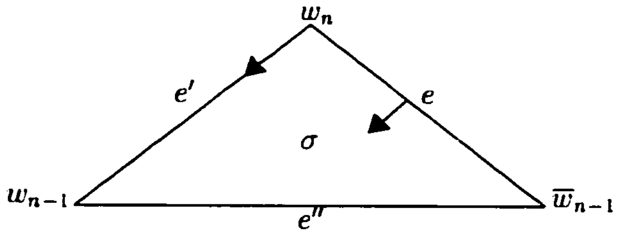

Figure 3.9: ??

A proof of this claim will establish the main theorem. We first show that W' is a discrete vector field.

We have defined W' so that if $W'(w) = e$ then $W'(e) = 0$. W' maps simplices of dimension two and higher to 0 so W' satisfies the property that if $\alpha \in \text{Im}(W')$ then $W'(\alpha) = 0$ for all $\alpha \in K$. Also by definition either $W'(\alpha) = 0$ or $W'(\alpha)$ is a codimension-1 face of α for all $\alpha \in K$. To verify that W' is a discrete vector field, all that remains to show is that for all $\alpha \in K$, if $\alpha \in \text{Im}(W')$ then there is a unique simplex γ with $W'(\gamma) = \alpha$. We recall that if $W'(\alpha) = \beta$ then α must be in the boundary of β. Consider an edge $e \in \text{Im}(W')$ and let w and w' denote the endpoints of e. If $W'(w) = e$ then there exists a combinatorial geodesic $w, e, w', \ldots, w^*$ in K from w to w^*, which implies $d_C(w', w^*) < d_C(w, w^*)$. Thus there does not exist a combinatorial geodesic $w', e, w, \ldots, w^*$, and $W'(w') \neq e$. Therefore for each edge $e \in \text{Im}(W')$ there is a unique vertex w with $W'(w) = e$. Next we need to show that if σ is a 2-simplex in $\text{Im}(W')$ then there is a unique edge e with $W'(e) = \sigma$.

Suppose that e is an edge and σ is a 2-simplex with $W'(e) = \sigma$. By lemma 28 the vertex opposite e is combinatorial distance $n - 1$ from w^*. If $m = n$ then two vertices of σ are combinatorial distance n from w^* and one vertex of σ is combinatorial distance $n-1$ from w^*. Lemma 28 then implies that e is the only face of σ satisfying $W(e) = \sigma$. If $m = n-1$, as in figure 3.4, then by corollary 20 σ is spanned by the three vertices w_n, w_{n-1}, and $\overline{w}_{n-1}$, so σ has two vertices w_{n-1} and $\overline{w}_{n-1}$ of combinatorial distance $n-1$ from w^*. w_{n-1} is opposite e in σ and $\overline{w}_{n-1}$ is opposite the edge e' spanned by w_n and w_{n-1}. But $W(w_n) = e'$ so $W(e') = 0$. In particular, $W(e') \neq \sigma$. By lemma 28, the edge e'' of σ different from e and e' can not map to σ since the vertex opposite e'' in K is combinatorially further from w^* than the endpoints of e''. Therefore there is exactly one edge e in K that satisfies $W(e) = \sigma$. We have proved that W is a discrete vector field on K.

By theorem 13, if W has no non-trivial closed W-paths then W is the gradient vector field of a discrete Morse function. Suppose $\alpha_0^0, \beta_0^1, \alpha_1^0, \ldots, \alpha_r^0, \beta_r^1, \alpha_{r+1}^0 = \alpha_0^0$ is a non-trivial closed W-path of vertices and edges in K. Then $d_C(\alpha_i, w^*) < d_C(\alpha_{i+1}, w^*)$. But then $d_C(\alpha_{r+1}, w^*) < d_C(\alpha_0, w^*) = d_C(\alpha_{r+1}, w^*)$, a contradiction. Therefore there are no non-trivial closed W-paths of vertices and edges.

Lemma 31. *Suppose W is a discrete vector field on the simplicial complex K, w^* a distinguished vertex of K, and $\alpha_0^1, \beta_0^2, \alpha_1^1, \ldots, \alpha_r^1, \beta_r^2, \alpha_{r+1}^1$ a W-path of edges and 2-simplices. For each edge α_i, denote its endpoints by y_i and y_i', and denote by x_i the vertex opposite α_i in β_i. If $d_C(y_i, w^*) = k$ and $d_C(y_i', w^*) = k-1$ then $d_C(x_{i+1}, w^*) = k-2$. If $d_C(y_i, w^*) = d_C(y_i', w^*) = k$ then $d_C(x_{i+2}, w^*) = k-2$.*

Proof. If $d_C(y_i, w^*) = k$ and $d_C(y_i', w^*) = k-1$ then by lemmas 20 and 28, $d_C(x_i, w^*) =$

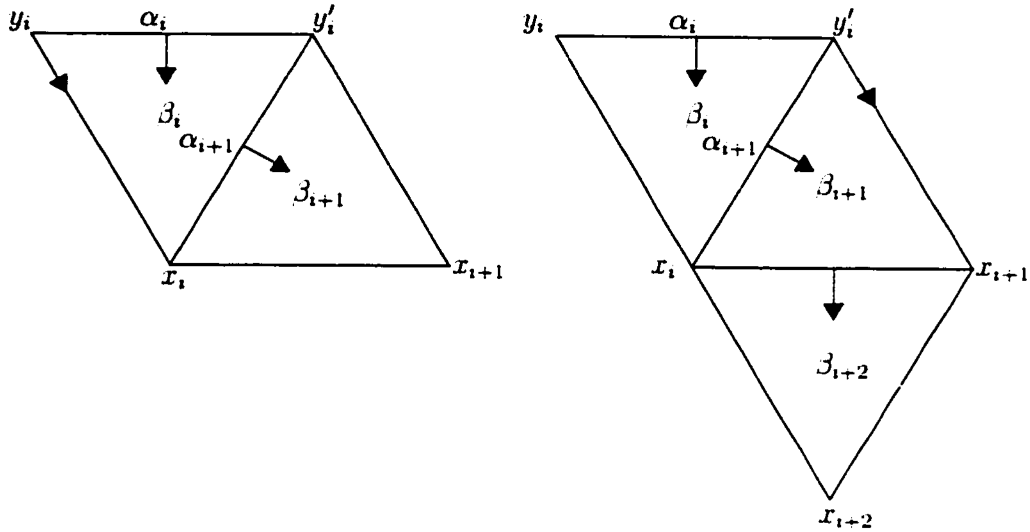

Figure 3.10: The two possibilities described in the proof of lemma 31

$k-1$ and W maps y_i to the edge spanned by y_i and x_i, implying that α_{i+1} must be the edge spanned by y_i' and x_i, both of which are distance $k-1$ from w^*. By lemma 28, $d_C(x_{i+1}, w^*) = k-2$.

If $d_C(y_i, w^*) = d_C(y_i', w^*) = k$ then by lemma 28, $d_C(x_i, w^*) = k-1$. Therefore α_{i+1} is spanned by vertices that are combinatorial distances k and $k-1$ from w^*. The previous case implies that $d_C(x_{i+2}, w^*) = k-2$. $\square$

Note that because the intersection of any two simplices must be a face of each of them, any closed W-path of edges and 2-simplices must contain at least three 2-simplices. Suppose that W has a non-trivial closed W-path $\alpha_0^1, \beta_0^2, \alpha_1^1, \ldots, \alpha_r^1, \beta_r^2, \alpha_{r+1}^1 = \alpha_0^1$ of edges and 2-simplices in K. Then x_r is equal to one of y_0 or y_0', and either $d_C(y_i, w^*) = k$ and $d_C(y_i', w^*) = k-1$ or $d_C(y_i, w^*) = d_C(y_i', w^*) = k$. In either case we have

$$k \leq d_C(x_r, w^*) < d_C(x_{r-2}, w^*) \leq d_C(x_0, w^*) = k - 1,$$

which is a contradiction. Therefore W has no nontrivial closed W-paths of edges and 2-simplices. Since W is only nonzero on vertices and edges, the discrete vector field W has no non-trivial closed W-paths and therefore by theorem 13, W is the gradient vector field of a discrete Morse function f.

The critical simplices of f are the simplices $\alpha \in W$ such that both $W(\alpha) = 0$ and $W \notin \mathrm{Im}(W)$. W is nonzero on every vertex of K except w^*, so w^* is the only critical vertex of K. W was defined so that every edge either maps to a 2-simplex or is in the image of W. Thus f is a discrete Morse function with exactly one critical vertex and no critical edges.

By the Morse inequalities on page 14 we have $\chi(K) = m_0 - m_1 + m_2 - m_3 = 1 - 0 + m_2 - m_3 = 1 + m_2 - m_3$. On the other hand, K is contractible so $\chi(K) = 1$ which means $m_2 = m_3$, or the number of critical 2-simplices equals the number of critical 3-simplices. The Morse complex for the Morse function V with coefficients in any field F is

$$\cdots \to 0 \to \mathcal{M}_3 \xrightarrow{\partial_3} \mathcal{M}_2 \xrightarrow{\partial_2} 0 \to < v^* > \to 0.$$

Since K is contractible, $0 = H_2(K, F) = \frac{ker \partial_2}{im \partial_3} = \frac{\mathcal{M}_2}{im \partial_3}$, which implies that the image of ∂_3 is $\mathcal{M}_2$. Thus ∂_3 is onto with coefficients in any field F. W is the gradient vector field associated to some discrete Morse function f. By theorem 14 from the section on Morse theory, if there exists a critical 3-simplex β and a critical 2-simplex

α with a unique gradient path from β to α, then K has a new Morse function which has the same critical simplices as f except that β and α are no longer critical.

Lemma 32. *If K is an n-complex satisfying the property that every $(n-1)$-simplex is a face of at most two n-simplices, then there are at most two gradient paths from any critical n-simplex to any critical $(n-1)$-simplex.*

Proof. Consider a combinatorial gradient path $\alpha_0^{n-1}, \beta_0^n, \alpha_1^{n-1}, \ldots, \alpha_r^{n-1}, \beta_r^n, \alpha_{r+1}^{n-1}$ in K. Observe that $\beta_i \neq \beta_{i+1}$ for $i = 0, \ldots, r-1$ since $f(\beta_i) > f(\alpha_{i+1}) \geq f(\beta_{i+1})$. α_i^{n-1} is the face of exactly two n-dimensional simplices, so if $V(\alpha_i) = \beta_i^n$ then β_{i-1} is uniquely determined in a pseudomanifold. For each β_i, α_i is uniquely determined since there exists at most $(n-1)$-simplex α_i with $V(\alpha_i) = \beta_i$. Thus given α_{r+1} and β_r, the gradient path is uniquely determined. Since α is the face of exactly two n-simplices, there are at most two gradient paths from $\partial\beta$ to α. □

We will show that there exists a critial 3-simplex β and a critical 2-simplex α with a unique gradient path from β to α. Let α be any critical 2-simplex. ∂_3 is onto with coefficients in any field F. Computing with $F = Z_2$, there exists a critical 3-simplex β with $< \partial_3\beta, \alpha >= 1 \bmod 2$. That is, mod 2 there is one gradient path from β to α. Computing with coefficients in $\mathbb{Z}$, this implies there is an odd number of gradient paths from β to α. By the previous lemma, the number of gradient paths from β to α is at most 2. Therefore there is a unique gradient path from β to α, and hence K has a new discrete Morse function which has the same critical simlices as f except that β and α are no longer critical. Continuing inductively, we conclude that there exists a discrete Morse function on K with exactly one critical vertex and no critical simplices of dimensions 1, 2, and 3. By theorem 7, K simplicially collapses to w^*, and theorem 29 is proved.

Chapter 4

A Combinatorial Isoperimetric Inequality

If every interior vertex v of a triangulated n-gon D satisfies deg $v \geq 6$ we say that D is an *admissible* **n**-*gon*. For $n \geq 3$ define V_n to be the maximum possible number of interior vertices of an admissible n-gon. Throughout this section, we write $\lfloor x \rfloor$ to denote the greatest integer that is less than or equal to x, and $\lceil x \rceil$ to denote the smallest integer that is greater than or equal to x. The goal of this chapter is to prove the following combinatorial isoperimetric inequality.

Theorem 33 (Combinatorial Isoperimetric Inequality). *Let V_n denote the maximum possible number of interior vertices of any triangulated n-gon D satisfying deg $v \geq 6$ for all interior vertices v of D. Then $V_n < \infty$ and V_n is given by the generating function*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{x^7 + (1-x)^2}{(1-x)^2(1-x^6)}.$$

An explicit formula for V_n is

$$V_n = \left\lfloor \frac{n^2 - 6n + 12}{12} \right\rfloor$$

Recall equation 3.1 for a triangulated n-gon all of whose interior vertices have degree at least six: $\sum_{ext\ v}(4 - \deg v) \geq 6$.

We will establish a recurrence relation for V_n, from which we obtain the generating function in the theorem. The first lemma establishes the smallest cases.

Lemma 34. $V_3 = V_4 = V_5 = 0$.

Proof. Equation 3.1 implies that every exterior vertex of an admissible 3-gon must have degree exactly two. Therefore an admissible 3-gon must be a triangle with no interior vertices, and $V_3 = 0$. Equation 3.1 implies that an admissible 4-gon must have at least one exterior vertex of degree two. This, together with the fact that $V_3 = 0$, implies that an admissible 4-gon must be two 2-simplices identified along an edge and hence can have no interior vertices. Therefore $V_4 = 0$. Again, equation 3.1 implies that an admissible 5-gon must have at least one exterior vertex of degree two. This implies that an admissible 5-gon must be a 2-simplex and a 4-gon identified along one exterior edge of each. $V_4 = 0$ then implies that an admissible 5-gon has no interior vertices, and therefore $V_5 = 0$. □

Lemma 35. *Let $n \geq 3$. Then $V_{n+1} \geq V_n$.*

Proof. An admissible n-gon with k interior vertices identified along an exterior edge with a 2-simplex forms an admissible $(n+1)$-gon with k interior vertices. This shows that $V_{n+1} \geq V_n$. □

Lemma 36. *Let $n \geq 4$. If $V_n < \infty$ then $V_{n+2} > V_n$.*

Proof. Suppose that V_n is finite and consider any admissible n-gon D with V_n interior vertices. We know from lemma 16 that D has an exterior vertex v with degree exactly 3. We will define a triangulated $(n+2)$-gon K that contains D as a subcomplex and such that v is an interior vertex of K. Let u_1 and u_5 be the two neighbors of v that are exterior vertices of D and let u_2, u_3, and u_4 be three distinct vertices not in D. Let σ_i be the 2-simplex spanned by u_i, u_{i+1}, and v for $i = 1, 2, 3, 4$. Let L be the simplicial complex consisting of $\sigma_1, \sigma_2, \sigma_3$, and σ_4, along with all of their faces. Then $K = D \cup L$ is a triangulated $(n+2)$-gon and every interior vertex of D is an interior vertex of K. Moreover, v is an interior vertex of K and a neighbor of u_2, u_3, and u_4 in L. $\deg_D v = 3$, hence v has degree 6 in K. Therefore K is an admissible $(n+2)$-gon with $V_n + 1$ interior vertices, which implies that $V_{n+2} > V_n$. □

We will need to establish a few more base cases before we can proceed with formulating the recurrence relation for V_n. Before doing this, however, we present the following three lemmas.

Lemma 37. *Suppose $V_n < \infty$ and let D be any admissible n-gon with V_n interior vertices. Then every exterior vertex v of D satisfies $\deg v \leq 4$.*

Proof. Suppose that v is an exterior vertex of D with $\deg v \geq 5$. Let u and u' be the exterior vertices of D that are neighbors of v and let v' be a point not in $|D|$. Let

σ_1 be the 2-simplex spanned by v, v', and u, σ_2 the 2-simplex spanned by v, v', and u', and L the simplicial complex consisting of σ_1 and σ_2 and all of their faces. The interior vertices of the triangulated n-gon $K = D \cup L$ are the interior vertices of D and v. v has degree at least six in K so K is an admissible n-gon. But then K is an admissible n-gon with $V_n + 1$ interior vertices, a contradiction since V_n is finite. □

Lemma 38. *Let D be an admissible n-gon. If D has $n-4$ consecutive exterior vertices of degree 3 then D has at most one interior vertex.*

Proof. Let $v_1, \ldots, v_n$ be the exterior vertices of D, labeled so that v_i is a neighbor of v_{i+1} for $i = 1, \ldots, n-1$. Suppose that the $n-4$ vertices $v_1, \ldots, v_{n-4}$ each have degree 3. This implies that v_n, v_1, and v_{n-3} all have a common neighbor $u \notin \{v_{n-3}, v_n\}$. Thus if u is an exterior vertex it must be either v_{n-1} or v_{n-2}. Then $V_3 = 0$ implies that D has no interior vertices. If u is an interior vertex then $v_{n-3}, v_{n-2}, v_{n-1}, v_n$, and u are the exterior vertices of a 5-gon which is a subcomplex of D. By lemma 34 the 5-gon has no interior vertices so u is the only interior vertex of D. □

We are now ready to prove the remaining three base cases for the induction proofs to follow.

Lemma 39. $V_6 = V_7 = 1$, *and* $V_8 = 2$.

Proof. Let D be any admissible 6-gon. If D has an exterior vertex of degree 2, then $V_5 = 0$ implies that D has no interior vertices. If D has no exterior vertices of degree 2 then equation 3.1 implies that all six exterior vertices of D must have degree exactly 3, resulting in one interior vertex. Hence $V_6 = 1$.

Let D be any admissible 7-gon. If D has an exterior vertex of degree 2, then $V_6 = 1$ implies that D has at most one interior vertex. If D has no exterior vertices of degree 2 then equation 3.1 again implies D must have at least six exterior vertices of degree 3. Six exterior vertices of degree 3 in a 7-gon means they are all consecutive, so by lemma 38, D has at most one interior vertex. A 7-gon where all seven exterior vertices have degree three shows that $V_7 = 1$.

Let D be any admissible 8-gon. Again, if D has an exterior vertex of degree 2 then $V_7 = 1$ implies that D has at most one interior vertex. If D has no exterior vertices of degree 2, then equation 3.1 implies that D has at least six exterior vertices of degree 3. If D has either seven or eight vertices of degree three then lemma 38 implies D has at most one interior vertex. However, we know that V_8 is at least two since by lemma 36, $V_8 > V_6 = 1$. The only remaining possibility is that the 8-gon has 6 exterior vertices of degree 3 and two exterior vertices of degree 4. Since $V_8 \geq 2$, lemma 38 implies an admissible 8-gon cannot have four consecutive exterior vertices of degree 3. Therefore there must be two groups of three consecutive exterior vertices of degree 3, separated by the two exterior vertices of degree 4. There is a unique admissible triangulation with this property, yielding an admissible 8-gon with two interior vertices. Therefore $V_8 = 2$. □

The next goal is to give an upper bound for V_n. We will do this by first finding an upper bound on the number of boundary neighbors of an admissible n-gon D. We can then look at the subcomplex of D whose exterior vertices are boundary neighbors of D and, using the following lemma, relate V_n to the number of vertices of this subcomplex.

Lemma 40. *Suppose $n_1, n_2, \ldots, n_m \geq 3$ and $m \geq 2$ are integers. If $n_1+n_2+\cdots+n_m = N$ and V_{n_i} is finite for $i = 1, \ldots, m$, then $V_{n_1} + V_{n_2} + \cdots + V_{n_m} < V_N$.*

Proof. The proof is by induction on m. Suppose $m = 2$. Let D_1 be an admissible n_1-gon and D_2 an admissible n_2-gon, containing V_{n_1} and V_{n_2} interior vertices, respectively. Identifying one exterior edge of D_1 with one exterior edge of D_2 yields a triangulated $(n_1 + n_2 - 2)$-gon with $V_{n_1} + V_{n_2}$ interior vertices, implying $V_{n_1} + V_{n_2} \leq V_{n_1+n_2-2}$. By lemma 36 implies that $V_{n_1+n_2-2} < V_{n_1+n_2}$. Therefore $V_{n_1} + V_{n_2} < V_{n_1+n_2} = V_N$.

For general m, by induction $V_{n_1} + V_{n_2} + \cdots + V_{n_{m-1}} < V_{n_1+n_2+\cdots+n_{m-1}}$. Therefore there exists an admissible $(n_1+n_2+\cdots+n_{m-1})$-gon D_1 containing $V_{n_1}+V_{n_2}+\cdots+V_{n_{m-1}}$ interior vertices. Let D_2 be an admissible n_m-gon containing V_{n_m} interior vertices. Identifying one exterior edge of D_1 with one exterior edge of D_2 yields an admissible $(n_1 + n_2 + \cdots + n_m - 2)$-gon with $V_{n_1} + V_{n_2} + \cdots + V_{n_m}$ interior vertices, implying $V_{n_1} + V_{n_2} + \cdots + V_{n_m} \leq V_{n_1+n_2+\cdots+n_m-2}$. By lemma 36, $V_{n_1+n_2+\cdots+n_m-2} < V_{n_1+n_2+\cdots+n_m}$. Therefore $V_{n_1} + V_{n_2} + \cdots + V_{n_m} < V_{n_1+n_2+\cdots+n_m} = V_N$. □

Lemma 41. *For $n \geq 7$, let D be an admissible n-gon with m distinct boundary neighbors. Then $m \leq n - 6$.*

Proof. Let $v_1, \ldots, v_n$ be the exterior vertices of D. Each exterior vertex v_i of D has at least two neighbors that are exterior vertices and at most deg $v_i - 2$ neighbors that are boundary neighbors of D. Each of the n pairs of neighboring exterior vertices has one common neighbor, which may or may not be a boundary neighbor of D, and is counted twice in the sum $\sum_{i=1}^{n}(\deg v_i - 2)$. Therefore $m \leq \sum_{i=1}^{n}(\deg v_i - 2) - n$ or equivalently, $\sum_{i=1}^{n} \deg v_i \geq m + 3n$. Equation 3.1 is equivalent to $\sum_{i=1}^{n} \deg v_i \leq 4n - 6$. Combining these two statements yields $m \leq n - 6$. $\square$

Corollary 42. $V_n < \infty$.

Proof. The proof is by induction on n. We have seen that V_n is finite for $n = 3, 4, \ldots, 8$. For the general case, suppose $n \geq 3$ and let D be any admissible triangulated $(n+6)$-gon. By lemma 41, the number of boundary neighbors of D is at most n. Let L be the set of simplices that contain an exterior vertex of D. The boundary neighbors of D are precisely the m exterior vertices of the simplicial complex $D - L$. The complex $D - L$ is a union of triangulated disks. By induction and lemma 40, the complex $D - L$ has at most $V_n < \infty$ interior vertices, and hence a total of at most $V_n + n$ vertices. Therefore D contains at most $V_n + n < \infty$ interior vertices. $\square$

In the proof of the preceeding corollary, we also proved the following statement.

Corollary 43. $V_{n+6} \leq V_n + n$ *for* $n \geq 3$.

The corollary gives an upper bound for V_n. To establish that $V_n + n$ is also a lower bound for V_{n+6}, we will give an example for each $n \geq 3$ of an admissible $(n+6)$-gon containing $V_n + n$ interior vertices. We define a sequence $\{D_n\}_{n \geq 3}$ of triangulated n-gons as follows. With the exception of D_7, the triangulated disks $D_3, \ldots, D_8$, pictured in figure 4, are the triangulated disks that we have described in the proofs of lemmas 34 and 39.

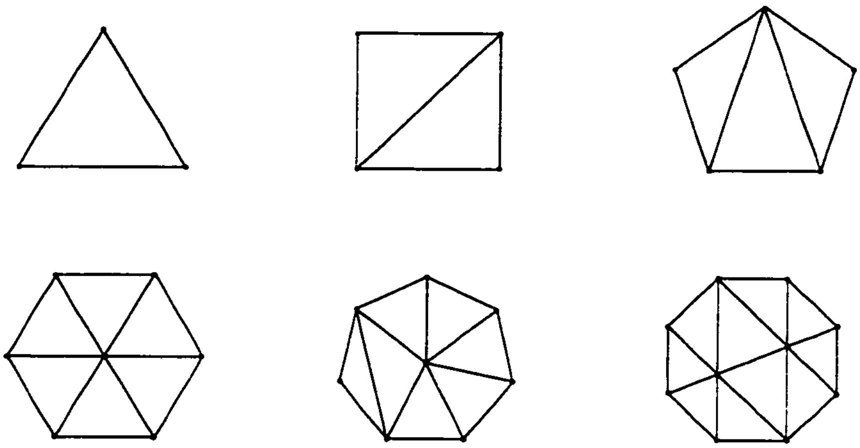

Figure 4.1: D_3, D_4, D_5, D_6, D_7, and D_8

For $3 \leq n \leq 8$, D_n is an admissible n-gon with V_n interior vertices, so D_n satisfies deg $v \leq 4$ for each exterior vertex v. Also note that in each of these cases deg $v = 6$ for every interior vertex v. We define the remaining terms of the sequence inductively.

Let $v_1, \ldots, v_n$ be the n distinct exterior vertices of D_n, labeled so that v_i is a neighbor of v_{i+1} for $i = 1, \ldots, n$. Consider a triangulated l-gon D_l that contains D_n as a subcomplex, where the l exterior vertices of D_l are precisely the vertices of D_l that are not vertices of D_n, and l is chosen to be the appropriate number so that v_i is a neighbor of $6 - \deg_{D_n} v_i$ consecutive exterior vertices of D_l, with the last neighbor of v_i equal to the first neighbor of v_{i+1} for each i, and the last neighbor of u_n is the first neighbor of u_1.

If $3 \leq n \leq 8$ this is possible since every exterior vertex of D_n has degree at most 4. If $n \geq 9$ this is possible because this construction process produces exterior vertices of degrees three and four. Also by construction, each interior vertex of D_l has degree

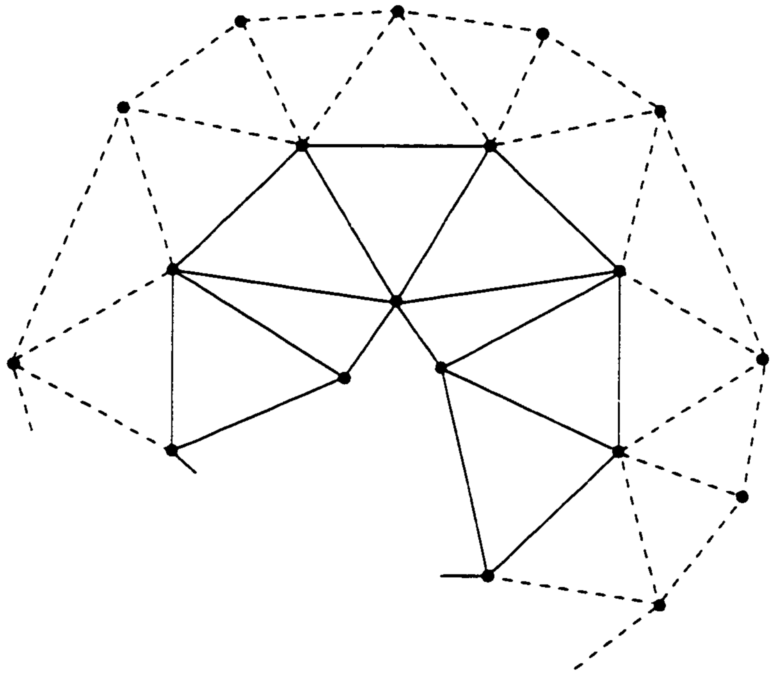

Figure 4.2: Constructing D_l from D_n

exactly 6 so equation 3.1 applies to give $\sum_{i=1}^{n}(4-\deg_{D_n} v_i) = 6$. Since two neighboring exterior vertices of D_n are joined to a common exterior vertex of D_l we have

$$\begin{aligned} l &= \sum_{i=1}^{n}(6 - \deg_{D_n} v_i) - n \\ &= n + \sum_{i=1}^{n}(4 - \deg_{D_n} v_i) \\ &= n + 6. \end{aligned}$$

Thus D_l is a triangulated $(n+6)$-gon. Define D_{n+6} to be the $(n+6)$-gon obtained from D_n in this manner.

Lemma 44. *For $n \geq 3$, D_{n+6} has $V_n + n$ interior vertices.*

Proof. By referring to lemmas 34 and 39 we can verify that for $3 \leq n \leq 8$, D_{n+6} contains $V_n + n$ interior vertices. If $n \geq 9$, then by induction D_n has $V_{n-6} + (n-6)$ interior vertices. By corollary 43, $V_n \leq V_{n-6} + (n-6)$, so the existence of D_n implies $V_n = V_{n-6} + (n-6)$. Therefore D_n contains V_n interior vertices. By construction, the interior vertices of D_{n+6} are the $V_n + n$ vertices of D_n. Therefore D_{n+6} contains $V_n + n$ interior vertices. □

Corollary 45. *For $n \geq 0$, V_n satisfies the recurrence relation $V_{n+6} = V_n + n$ with the convention that $V_0 = 1$ and $V_1 = V_2 = 0$.*

Proof. For $n \geq 3$ we have already seen that $V_{n+6} \leq V_n + n$. The previous lemma shows that $V_{n+6} \geq V_n + n$, establishing the corollary for $n \geq 3$. The lemma is easily verified for $n = 0, 1, 2$. □

To solve this recurrence relation, we let $G(x) = \sum_{n=0}^{\infty} V_n x^n$. Taking sums of each quantity in the recurrence relation we obtain $\sum_{n=0}^{\infty} V_{n+6}x^n = \sum_{n=0}^{\infty} V_n x^n + \sum_{n=0}^{\infty} nx^n$. Then

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} V_n x^n \\
&= \sum_{n=0}^{\infty} V_{n+6}x^n - \sum_{n=0}^{\infty} nx^n \\
&= \frac{1}{x^6}\sum_{n=0}^{\infty} V_{n+6}x^{n+6} - x\sum_{n=0}^{\infty} nx^{n-1} \\
&= \frac{1}{x^6}\left[G(x) - V_0 - V_1 x - \cdots - V_5 x^5\right] - x \cdot \frac{d}{dx}\sum_{n=0}^{\infty} x^n \\
&= \frac{1}{x^6}\left[G(x) - 1\right] - x \cdot \frac{d}{dx}\frac{1}{1-x} \\
&= \frac{1}{x^6}\left[G(x) - 1\right] - \frac{x}{(1-x)^2}
\end{aligned}$$

Solving for $G(x)$ yields

$$G(x) = \frac{x^7 + (1-x)^2}{(1-x)^2(1-x^6)}.$$

which is the generating function in theorem 33.

In order to find an explicit formula for V_n we write

$$\begin{aligned}
G(x) &= \frac{1}{1-x^6}\left(1 + x^7 \cdot \frac{1}{(1-x)^2}\right) \\
&= \sum_{n=0}^{\infty} x^{6n}\left(1 + x^7 \sum_{n=0}^{\infty} nx^{n-1}\right) \\
&= \sum_{n=0}^{\infty} x^{6n}\left(1 + \sum_{n=0}^{\infty} nx^{n+6}\right)
\end{aligned}$$

The sequences

$$\begin{aligned}\{a_n\} &= \{1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,\ldots \text{ and}\\ \{b_n\} &= \{1,0,0,0,0,0,0,1,2,3,4,5,6,7,8,9,10,11,12\ldots\end{aligned}$$

are the sequences of the coefficients for the generating functions $A(x) = \sum_{n=0}^{\infty} x^{6n}$ and $B(x) = 1 + \sum_{n=0}^{\infty} nx^{n+6}$. Thus the sequence $\{V_n\}$ is obtained by convolution of $\{a_n\}$ and $\{b_n\}$. From these sequences we see that V_n is the sum of every sixth term of $\{b_n\}$, beginning with b_n and moving backwards in the sequence. $b_0 = 1$ contributes an extra 1 whenever n is a multiple of 6. Therefore,

$$V_n = \begin{cases} \sum_{i=1}^{\lfloor \frac{n}{6} \rfloor} (n-6i) & \text{if } n \text{ is not a multiple of } 6 \\ \sum_{i=1}^{\lfloor \frac{n}{6} \rfloor} (n-6i) + 1 & \text{if } n \text{ is a multiple of } 6 \end{cases}$$

Simplifying gives $\sum_{i=1}^{\lfloor \frac{n}{6} \rfloor} (n-6i) = \lfloor \frac{n}{6} \rfloor \left(n - 3 - 3\lfloor \frac{n}{6} \rfloor\right)$. Therefore

$$V_n = \begin{cases} \lfloor \frac{n}{6} \rfloor \left(n - 3 - 3\lfloor \frac{n}{6} \rfloor\right) & \text{if } n \text{ is not a multiple of } 6 \\ \lfloor \frac{n}{6} \rfloor \left(n - 3 - 3\lfloor \frac{n}{6} \rfloor\right) + 1 & \text{if } n \text{ is a multiple of } 6 \end{cases} \tag{4.2}$$

Lemma 46.

$$\left\lfloor \frac{n}{6} \right\rfloor \left(n - 3 - 3\left\lfloor \frac{n}{6} \right\rfloor\right) = \begin{cases} \left\lfloor \frac{n^2-6n+12}{12} \right\rfloor & \textit{if } n \textit{ is not a multiple of } 6 \\ \frac{n^2-6n}{12} & \textit{if } n \textit{ is a multiple of } 6 \end{cases}$$

Proof. First suppose that n is a multiple of 6. Then $\lfloor \frac{n}{6} \rfloor = \frac{n}{6}$ so we have

$$\begin{aligned} \left\lfloor \frac{n}{6} \right\rfloor \left(n-3-3\left\lfloor \frac{n}{6} \right\rfloor\right) &= \frac{n}{6}\left(n-3-3\cdot\frac{n}{6}\right) \\ &= \frac{n^2-6n}{12} \end{aligned}$$

as desired.

Now suppose that n is not a multiple of 6. Then $6m < n < 6(m+1)$ for some integer m. Observe that for any two integers r and s, $6r < s < 6(r+1)$ if and only if $\lfloor \frac{s}{6} \rfloor = r$ and $\lceil \frac{s}{6} \rceil = r+1$. Hence $\lfloor \frac{n}{6} \rfloor = m$. In addition,

$$\left\lfloor \frac{n}{6} \right\rfloor \left(n-3-3\left\lfloor \frac{n}{6} \right\rfloor\right) = \left\lceil \frac{n^2-6n}{12} \right\rceil \tag{4.3}$$

if and only if

$$6\left(\left\lfloor \frac{n}{6} \right\rfloor \left(n-3-3\left\lfloor \frac{n}{6} \right\rfloor\right)-1\right) < \frac{n^2-6n}{2} < 6\left(\left\lfloor \frac{n}{6} \right\rfloor \left(n-3-3\left\lfloor \frac{n}{6} \right\rfloor\right)\right) \tag{4.4}$$

Let a be any real number and consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - (6+12a)x + 36a + 36a^2$. f is concave up and has roots $6a$ and $6(a+1)$. Therefore $f(x) < 0$ if and only if $6a < x < 6(a+1)$.

Now because $6m < n < 6(m+1)$, this implies that $n^2 - (6+12m)n + 36m + 36m^2 <$

0. Equivalently,

$$\begin{aligned}
n^2 - 6n &< 12mn - 36m - 36m^2 \\
&= 12m(n - 3 - 3m) \\
&= 12\left(\left\lfloor\frac{n}{6}\right\rfloor\left(n - 3 - 3\left\lfloor\frac{n}{6}\right\rfloor\right)\right)
\end{aligned}$$

and hence

$$\frac{n^2 - 6n}{2} < 6\left(\left\lfloor\frac{n}{6}\right\rfloor\left(n - 3 - 3\left\lfloor\frac{n}{6}\right\rfloor\right)\right).$$

It remains to show that

$$6\left(\left\lfloor\frac{n}{6}\right\rfloor\left(n - 3 - 3\left\lfloor\frac{n}{6}\right\rfloor\right) - 1\right) < \frac{n^2 - 6n}{2}.$$

Consider the function $g(x) = f(x) + 12$. g has no real roots and $g(0) = 36a + 36a^2 + 12 > 0$ since the polynomial $p(x) = 36x^2 + 36x + 12$ has no real roots and is positive at $x = 0$. Thus $g(x) > 0$ for all x. Setting $a = m$ gives $g(n) = n^2 - (6 + 12m)n + 36m + 36m^2 + 12 > 0$. Equivalently,

$$\begin{aligned}
n^2 - 6n &> 12mn - 36m - 36m^2 - 12 \\
&= 12(m(n - 3 - 3m) - 1) \\
&= 12\left(\left\lfloor\frac{n}{6}\right\rfloor\left(n - 3 - 3\left\lfloor\frac{n}{6}\right\rfloor\right) - 1\right).
\end{aligned}$$

and hence

$$\frac{n^2-6n}{2} > 6\left(\left\lfloor\frac{n}{6}\right\rfloor\left(n-3-3\left\lfloor\frac{n}{6}\right\rfloor\right)-1\right),$$

as desired. Thus we have shown equation 4.4 to be true, which implies that equation 4.3 holds.

In order to prove lemma 46, it remains to show that in the case that n is not a multiple of 6, we have

$$\left\lceil\frac{n^2-6n}{12}\right\rceil = \left\lfloor\frac{n^2-6n+12}{12}\right\rfloor.$$

Observe that if n is not a multiple of 6 then $n(n-6)$ can not be a multiple of 12. To see this, suppose that n is not a multiple of 6. Then either 2 does not divide n or 3 does not divide n. If 2 does not divide n then $n(n-6)$ is odd and hence not a multiple of 12. If 3 does not divide n then 3 also does not divide $n-6$, and again $n(n-6)$ can not be a multiple of 12. Note that if r is a real number then r is not an integer if and only if $\lceil r\rceil = \lfloor r+1\rfloor$. Since n not a multiple of 6 implies $n(n-6)$ not a multiple of 12, $\left\lfloor\frac{n^2-6n}{12}\right\rfloor$ is not an integer and therefore

Therefore

$$\left\lceil\frac{n^2-6n}{12}\right\rceil = \left\lfloor\frac{n^2-6n}{12}+1\right\rfloor = \left\lfloor\frac{n^2-6n+12}{12}\right\rfloor$$

and the proof is complete. □

Finally, note that if n is a multiple of 6 then

$$\frac{n^2-6n}{12}+1 = \left\lfloor\frac{n^2-6n}{12}+1\right\rfloor = \left\lfloor\frac{n^2-6n+12}{12}\right\rfloor.$$

Putting this together with equation 4.1 and lemma 46 yields the final expression for V_n for all $n \geq 3$:

$$V_n = \left\lfloor \frac{n^2 - 6n + 12}{12} \right\rfloor,$$

and the proof of the theorem is complete.

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