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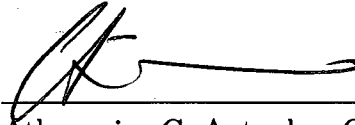
**Passivity preserving model reduction in the
context of spectral zero interpolation**

by

Roxana Ionutiu

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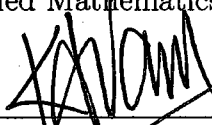
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ABSTRACT

Passivity preserving model reduction in the context of spectral zero interpolation

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This thesis presents a new passivity preserving model reduction method for circuit simulation, based on interpolation of dominant spectral zeros. Implemented as an eigenvalue approximation problem, the dominant spectral zero method (dominant SZM) is based on the subspace accelerated dominant pole algorithm (SADPA), which computes dominant spectral zeros automatically. The application of dominant SZM is extended beyond its interpolatory nature, proposing solutions for several problems in passive reduction. In particular, better approximation is achieved when combined with partial realization for descriptor systems, a framework for SISO reduction of the voltage transfer function in transmission lines is presented, and the implementation of MIMO dominant SZM is developed. Dominant SZM reduces automatically passive circuits irrespective of how the system equations are formulated, transmission line models with controlled sources, or circuits containing susceptance elements. Results show that dominant SZM gives comparable and often more accurate reduced models than state of the art techniques.

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Chapter 1

Introduction

1.1 Motivation

Model reduction for large-scale systems arising in circuit simulation has been extensively studied. See for instance [1] for a selection of recent results. Model reduction of transmission lines is also addressed in [2]. Such circuits yield systems with millions of internal variables, making simulation in full dimension unfeasible. A model of lower dimension is needed to approximate the behavior of the original circuit and replace it during simulation. For interconnects involving resistors, capacitors, inductors or controlled sources, the associated systems are *passive*, with *positive real* transfer functions (for details see [3]). We are interested in reduced order models that are also passive. This guarantees stability of the overall nonlinear macro-model, when the reduced model replaces the original in the simulation process [4]. Furthermore, passivity insures that the reduced system is realizable, meaning that it can be synthesized into an electrical circuit with RLC components. This would be used in simulation in place of the original circuit [5]. The reduction method should have an efficient, robust implementation, suitable for large-scale applications.

This thesis proposes a passivity-preserving model order reduction method that is more general than existing methods such as PRIMA [4, 6]. The *dominant spectral zero interpolation method (dominant SZM)* puts no constraints on the system matrices and generates passive reduced order models automatically. Based on an iterative

solver for an associated Hamiltonian eigenvalue problem [7, 8], the method uses a rational Krylov approach to construct a reduced model via interpolation. Unlike modal approximation [9, 10, 11], which matches dominant poles, dominant SZM matches the dominant *spectral zeros* of the original system, guaranteeing passivity. In comparison to other passivity preserving methods such as PRBT [12], it avoids costly large-scale matrix factorizations.

Passivity preserving model reduction via spectral zero interpolation was first introduced in [13, 14], however several open questions remained. This thesis proposes solutions to these problems and incorporates them in the reduction method as follows:

- (P1) In the spectral zero method (SZM) of [13, 14], it was still unclear which spectral zeros to select as interpolation points, such that the reduced system models closely the behavior of the original. Here a dominance criterion for selecting the spectral zeros is proposed.
- (P2) The second issue arose from large scale applications, where computing all spectral zeros is computationally infeasible. An iterative algorithm is required to compute automatically only the spectral zeros of interest. Using the recent work of [11], [15, Chapter 3], a numerically efficient algorithm is proposed for computing the *dominant spectral zeros*, which is suitable for large scale problems.
- (P3) For systems with $\mathbf{D} = \mathbf{0}$ (see (1.1)), the projection method in [13, 14] is modified, so that the reduced system is strictly passive.

Aside from the development of dominant SZM, several new approaches for passivity preserving reduction are presented:

- (P4) A framework is developed for combining dominant spectral zero interpolation with matching of Markov parameters for descriptor systems arising in circuit

simulation. The approach enhances the quality of the approximation and is one alternative for addressing (P3).

- (P5) A procedure for reducing a single-input-single-output network with respect to the voltage transfer function is described, which insures that the reduced voltage transfer function is underlying a reduced model that is realizable with RLC components.
- (P6) Reduction of multiple-input-multiple-output (MIMO) systems with dominant SZM is implemented and accompanied by examples.

The rest of the thesis is organized as follows. As part of this introductory chapter, Sect. 1.2 sets the general framework for model reduction by projection, reviews definitions and properties for descriptor systems, and introduces passivity. Sect. 1.3 discusses possible challenges for passivity preservation with some popular reduction techniques. Chapter 2* describes in detail the dominant spectral zero interpolation method, *dominant SZM*. Problems (P1,P2,P3) are addressed in that chapter, and the method is compared with other popular techniques (PRIMA, PRBT, modal approximation) via extensive numerical results. Further developments based on dominant SZM follow in Chapters 3 and 4, where (P4) and (P5, P6) are addressed respectively. The contributions of this thesis are summarized in Chapter 5.

1.2 Background on model reduction

This section formalizes the model reduction problem mathematically. It introduces basic definitions and concepts used throughout the thesis and in model reduction

*This chapter is part of an individual article [16].

problems in general. Established results for model reduction preserving stability and passivity are also reviewed. The background provided here is by no means comprehensive, and the reader is referred to [3] for further details and proofs.

1.2.1 The model reduction problem

The general framework in model reduction involves the approximation of an original dynamical system described by a set of differential algebraic equations (DAEs). Using system-theoretic notation, these equations are frequently in the *descriptor-form*[†] $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ [3, 14, 17], [15, Chapter 1] namely:

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \quad (1.1)$$

where the entries of $\mathbf{x}(t)$ are the system's internal variables, $\mathbf{u}(t)$ is the system input and $\mathbf{y}(t)$ is the system output, with dimensions $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$. Correspondingly, $\mathbf{E} \in \mathbb{R}^{n \times n}$, $\mathbf{A} \in \mathbb{R}^{n \times n}$, $(\mathbf{A}, \mathbf{E})$ is a regular pencil (see definition 1.2), $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, $\mathbf{D} \in \mathbb{R}^{p \times m}$. The *dimension* n of Σ is usually very large.

The goal of system approximation is to constrain the dynamics of the system to lie in a lower dimensional subspace $k \ll n$. The associated reduced order model $\hat{\Sigma}(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \mathbf{D})$ is:

$$\begin{cases} \hat{\mathbf{E}}\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \quad (1.2)$$

where $\hat{\mathbf{x}} \in \mathbb{R}^k$, $\hat{\mathbf{E}} \in \mathbb{R}^{k \times k}$, $\hat{\mathbf{A}} \in \mathbb{R}^{k \times k}$, $\hat{\mathbf{B}} \in \mathbb{R}^{k \times m}$, $\hat{\mathbf{C}} \in \mathbb{R}^{p \times k}$, $\mathbf{D} \in \mathbb{R}^{p \times m}$.

[†]The term *descriptor system* denotes systems with $\mathbf{E}$ possibly singular, as opposed to *state space system* where $\mathbf{E}$ is invertible or the identity matrix.

Model reduction methods considered in this thesis belong to the class of *projection methods*, in which the reduced system $\hat{\Sigma}(\hat{E}, \hat{A}, \hat{B}, \hat{C}, D)$ is obtained by projecting the internal variables of the original system $\mathbf{x}$ onto the subspace $ColSpan \mathbf{V} \subset \mathbb{R}^{n \times k}$, along $Null \mathbf{W}^* \subset \mathbb{R}^{k \times n}$, where $\mathbf{V}$ and $\mathbf{W}$ need to be computed. Denoting the internal variables of the reduced order model with $\hat{\mathbf{x}}$, the approximate internal variables of the original system become $\mathbf{x} \simeq \mathbf{V}\hat{\mathbf{x}}$, and the reduced system matrices are:

$$\hat{E} = \mathbf{W}^* \mathbf{E} \mathbf{V}, \quad \hat{A} = \mathbf{W}^* \mathbf{A} \mathbf{V}, \quad \hat{B} = \mathbf{W}^* \mathbf{B}, \quad \hat{C} = \mathbf{C} \mathbf{V}. \quad (1.3)$$

The associated *oblique projection* which reduces the system is $\Pi = \overline{\mathbf{V}} \overline{\mathbf{W}}^*$, obtained via an *LU* factorization: $\mathbf{W}^* \mathbf{E} \mathbf{V} = \mathbf{L} \mathbf{U}$, $\overline{\mathbf{V}} = \mathbf{V} \mathbf{U}^{-1}$, $\overline{\mathbf{W}}^* = \mathbf{L}^{-1} \mathbf{W}^*$, so that $\overline{\mathbf{W}}^* \mathbf{E} \overline{\mathbf{V}} = \mathbf{I}_k$.

Model reduction methods differ in the way the corresponding projecting matrices $\mathbf{V}$ and $\mathbf{W}$ are constructed. These determine how accurately $\hat{\Sigma}$ approximates Σ , as well as what properties of the original system Σ are preserved by the reduced $\hat{\Sigma}$. In circuit simulation, preserving the *stability* and *passivity* (defined in Sect. 1.2.2) of the original system is of crucial importance. The method under consideration achieves this through appropriate choice of $\mathbf{V}$ and $\mathbf{W}$.

1.2.2 System parameters

In the following, a descriptor system $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ described by (1.1) is given. $\mathbf{U}(s)$ and $\mathbf{Y}(s)$ are the Laplace transforms of the system's input $\mathbf{u}(t)$ and output $\mathbf{y}(t)$ respectively.

Definition 1.1 The system transfer function is defined as:

$$\mathbf{H}(s) := \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + D$$

Definition 1.2 The poles of Σ are all $s \in \mathbb{C}$ where $\mathbf{H}(s) = \infty$, i.e., the generalized eigenvalues of the pair $(\mathbf{A}, \mathbf{E})^\dagger$. Throughout this work, it is assumed that the matrix $(\mathbf{A} - \sigma\mathbf{E})$ is nonsingular for some $\sigma \in \mathbb{C}$ [17], which in mathematical terms is expressed as the *matrix pencil* $(\mathbf{A}, \mathbf{E})$ being *regular*.

Corollary 1.1

Σ is stable if for all poles λ , with $|\lambda| \neq \infty$, $\text{Re}(\lambda) < 0$ is satisfied, i.e., if all finite poles are located in the left half of the complex plane.

Definition 1.3 The spectral zeros of Σ are defined as:

$$\text{All } s \in \mathbb{C} \text{ such that } \det[\mathbf{H}(s) + \mathbf{H}^*(-s)] = 0, \text{ where} \quad (1.4)$$

$$\mathbf{H}^*(-s) = \mathbf{B}^*(-s\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^* + \mathbf{D}^*. \quad (1.5)$$

Definition (1.4) reveals the symmetry of spectral zeros. Every spectral zero s_i has a mirror image in the complex plane with respect to the imaginary axis. Real spectral zeros s_i come in pairs $(s_i, -s_i)$ while complex spectral zeros come in quadruples of the form:

$$(s_i, s_i^*, -s_i, -s_i^*) \quad (1.6)$$

[†]For $\mathbf{A} \in \mathbb{C}^{n \times n}$, $\mathbf{E} \in \mathbb{C}^{n \times n}$, λ is a generalized eigenvalue of $(\mathbf{A}, \mathbf{E})$, if $\exists \mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{E}\mathbf{x}$.

1.2.3 Passivity and positive realness

Passive systems (such as RLC circuits) are systems which do not generate energy. They are *dissipative* with respect to the supply rate (1.7) (think of power delivered to the system, through the external variables input $\mathbf{u}$ and output $\mathbf{y}$). Dissipativity w.r.t. $\mathbf{s}$ means that the system absorbs energy (supply) [3]. In other words, passive systems satisfy the dissipation inequality (1.8) [3, Theorem 5.22].

$$\mathbf{s} = \mathbf{y}^* \mathbf{u} + \mathbf{u}^* \mathbf{y} \quad (1.7)$$

$$\operatorname{Re} \left[\int_{-\infty}^t \mathbf{u}^*(\tau) \mathbf{y}(\tau) d\tau \right] \geq 0 \quad \forall t \in \mathbb{R} \quad (1.8)$$

Positive real rational functions $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ on the other hand satisfy the following conditions:

- $\mathbf{H}(s)$ is analytic for $\operatorname{Re}(s) > 0$
- $\operatorname{Re}[\mathbf{H}(s)] \geq 0$ for $\operatorname{Re}(s) \geq 0$, where s is not a pole of $\mathbf{H}(s)$.

The following well-known theorem relates passivity to the positive realness of the underlying transfer function. Corollary 1.2 is the foundation for the spectral zero interpolation approach for preserving passivity [13, 14].

Theorem 1.1

$\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is passive iff $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ is positive real.

Corollary 1.2

If for all spectral zeros s we have $\operatorname{Re}(s) \neq 0$, then Σ is strictly passive.

1.3 Challenges for passivity preservation

Many model reduction methods have been developed which preserve passivity, but they rely on some restrictive assumptions about the system matrices. This section summarizes such possible hurdles. PRBT [12] for instance requires an invertible $\mathbf{E}$, while PRIMA [4] starts from system equations in a so-called “*passive form*” [5]. Dominant SZM on the other hand imposes none of these restrictions. Nevertheless, the challenge for dominant SZM is ensuring strict passivity for systems with $\mathbf{D} = 0$, and possible solutions are presented in this thesis.

1.3.1 $\mathbf{E}$ singular and PRBT

In circuit simulation applications, systems with singular $\mathbf{E}$ arise frequently and provide a challenge to the model reduction problem formulated in Sect. 1.2.1, because some of the associated system poles and spectral zeros (defined in Sect. 1.2.2) are at infinity. Model reduction thus becomes numerically more sensitive and constructing the required projections is more involved than for the case when $\mathbf{E}$ is invertible. A study of different model reduction methods for systems involving an invertible $\mathbf{E}$ can be found in [18]. If $\mathbf{E}$ is invertible, system (1.1) can be converted to state-space representation through left-multiplication by $\mathbf{E}^{-1}$. See [3] for an overview of model reduction methods for state-space systems.

Some reduction methods have been adapted for general descriptor systems with singular $\mathbf{E}$. *Balanced truncation* for descriptor systems is described in [17], and *modal approximation* as presented in [11, 19] also handles descriptor systems. For Krylov methods we refer to [20]. However, these methods are not passivity preserving in general. PRBT [12] is passivity preserving, but relies on invertible $\mathbf{E}$. Dominant SZM on the other hand is passivity preserving and deals easily with singular $\mathbf{E}$.

1.3.2 “Non-passive” system formulation and PRIMA

In circuit simulation, the system equations are often expressed in the *modified nodal analysis (MNA)* representation (1.9).

$$\begin{cases} \mathcal{C}\dot{\mathbf{x}}(t) + \mathcal{G}\mathbf{x}(t) = \mathcal{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathcal{L}\mathbf{x}(t) \end{cases} \quad (1.9)$$

For MNA equations (1.9), internal variables are chosen as the node voltages and loop currents: $\mathbf{x}(t) = [\mathbf{v}(t), \mathbf{i}(t)]^T$. In this case (1.1) becomes MNA (1.9), with $\mathbf{E} = \mathcal{C}$, $\mathbf{A} = -\mathcal{G}$, $\mathbf{B} = \mathcal{B}$, $\mathbf{C} = \mathcal{L}$, $\mathbf{D} = 0$. To preserve passivity via congruence transformations, industry preferred methods such as PRIMA [4] assume systems are in “passive-form”, imposing several constraints on the system matrices:

$$\mathcal{C} \text{ is definite} \quad (1.10)$$

$$\mathcal{G} + \mathcal{G}^* \text{ is definite} \quad (1.11)$$

$$\mathcal{L} = \mathcal{B}^*. \quad (1.12)$$

In applications however, some circuit simulators are known to generate system equations which do not satisfy these assumptions. In [21], for instance, the authors point out difficulties in automating the model reduction of a transmission line, for which (1.10) is violated. In such circumstances, a sign change which negates current variables is needed before applying a reduction algorithm like PRIMA, so that $\mathcal{C}$ in (1.9) becomes definite. This reorganization of equations not only requires knowledge of how they are generated internally, but may also conflict with other simulation types or data-structures. Dominant SZM avoids these manipulations by reducing a system

with an indefinite $\mathbf{E}$ in (1.1) directly, while preserving stability and passivity. Other examples which violate (1.11) include circuits with controlled sources and circuits with susceptance elements (RCS circuits) [22, 23]. Unlike with PRIMA, passivity is guaranteed when reducing such systems with dominant SZM. Finally, with dominant SZM, (1.12) need not hold to insure passivity for the reduced model, as long as the original transfer function is positive real [3, 13, 14].

To accommodate reduction for a broader class of dynamical systems, the descriptor system notation (1.1) is used throughout this thesis, where $\mathbf{E}$ is possibly indefinite and singular, and no assumptions are made about the other system matrices unless otherwise stated (see Chapter 3). The dominant spectral zero interpolation method proposed in this thesis successfully handles descriptor systems while preserving passivity and stability, irrespective of whether (1.10), (1.11), (1.12) hold, and can be implemented efficiently.

1.3.3 $\mathbf{D} = 0$ and dominant SZM

Often in circuit simulation, the original differential equations are formulated so that $\mathbf{D} = 0$. The spectral zero interpolation approach as developed in [13, 14] assumed $\mathbf{D} \neq 0$ in the output equation of the original system, so that the reduced model is strictly passive. This thesis extends the application of dominant SZM to guarantee stable and strictly passive reduced models also when $\mathbf{D} = 0$. Possible approaches are treated in Sect. 2.3.2 and Chapter 3. The method in Sect. 2.3.2 makes no assumptions about the structure of the system matrices. In Chapter 3, information about the structure is used.

Chapter 2

Passivity preserving model reduction using dominant spectral zero interpolation

We present a new approach for passivity preserving reduction via interpolation of spectral zeros. A dominance criterion for spectral zero selection is proposed, which enhances approximation quality. Spectral zeros are computed as eigenvalues of an associated Hamiltonian eigenvalue problem. This is solved efficiently using the subspace accelerated dominant pole algorithm (SADPA), which computes relevant spectral zeros automatically. The passive reduced order model is computed by projection. Our method extends beyond applications suitable for reduction via PRIMA: it reduces automatically systems with an underlying possibly indefinite $\mathbf{E}$ matrix, transmission line models with controlled sources, or circuits containing susceptance elements. We provide approximation results with the proposed method, and compare it against state of the art techniques such as modal approximation, PRIMA and positive real balanced truncation (PRBT).

2.1 Overview

This chapter describes the complete procedure for passivity preserving reduction via interpolation of dominant spectral zeros. It outlines model reduction by projection with efficient dominant spectral zero interpolation in three stages: construction of passivity preserving projection (Sect. 2.2.1), dominant spectral zero selection (Sect. 2.2.2) and implementation with the recent subspace accelerated dominant pole algorithm

(SADPA) (Sect. 2.2.3). Sect. 2.3 addresses possible formulations for the Hamiltonian eigenvalue problem used in SADPA. Numerical results are presented in Sect. 2.4, followed by a discussion on complexity and convergence for dominant SZM in Sect. 2.5. Finally, Sect. 2.6 summarizes the analysis and motivates further research.

2.2 Dominant Spectral Zero Method with SADPA

This section addresses two of the open problems stated in Sect. 1.1: (P1) a classification of spectral zeros that reveals which ones to pick as interpolation points, and (P2) their efficient computation using SADPA [11], [15, Chapter 3] ((P3) is treated in Sect. 2.3.2). The method combines three ingredients: a dominance criterion for spectral zeros, a rational Krylov-type interpolation which guarantees passivity, and an iterative solver of a large-scale Hamiltonian eigenvalue problem. These make SZM suitable for reducing passive large-scale systems with more general structures and no constraints on the system matrices.

The proposed dominance criterion for the selection of spectral zeros is related to the magnitude of the associated *residues*. Dominant spectral zeros, similar to dominant poles [11], [15, Chapter 3], characterize the system's response, thus using them as interpolation points enhances approximation quality. Because most dominant spectral zeros can be scattered anywhere in the complex plane, computing them efficiently is a demanding task. We overcome this by first converting the spectral zero computation into a generalized Hamiltonian eigenvalue problem, similar to [14] (see also [7] and [8] for numerical methods involving Hamiltonian eigenvalue problems). Dominant spectral zeros and the associated projection eigenvectors are then computed efficiently using a variant of the *Subspace Accelerated Dominant Pole Algorithm (SADPA)* (introduced in [11] and refined in [15, Chapter 3]). SADPA is able

to find iteratively spectral zeros and associated residues, according to a dominance criterion specified a priori, meeting the purpose of efficient spectral zero computation for large-scale applications.

2.2.1 Model reduction by projection preserving passivity

The spectral zero method (SZM) belongs to the class of Krylov based reduction methods. These exploit the use of Krylov subspace iterations to achieve system approximation by *moment matching* [3]. In the general case, using the *rational Krylov* approach [20], [24], reduced systems are obtained which match moments at preassigned *interpolation points* in the complex plane [3]. SZM is a rational Krylov method, where interpolation points are a subset of the spectral zeros of the original system [13], [14]. This selection guarantees stability and passivity for the reduced system [13], [14].

Summarizing [13] and [14], we describe how to build projection matrices $\mathbf{V}$ and $\mathbf{W}$ such that the reduced system $\hat{\Sigma}(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \mathbf{D})$ is strictly passive (i.e., $\hat{\Sigma}$ has no spectral zeros on the $j\omega$ axis) and stable (i.e., all poles of $\hat{\Sigma}$ are located in the left half of the complex plane).

From (1.4), spectral zeros are poles of the rational function:

$$\mathbf{G}(s) = [\mathbf{H}(s) + \mathbf{H}^*(-s)]^{-1}, \quad (2.1)$$

where $\mathbf{H}^*(-s)$ is computed as in (1.5). We seek the form of the system $\Sigma_{\mathbf{G}}$, associated with transfer function (2.1). This allows us to compute spectral zeros of the original system Σ as poles of $\Sigma_{\mathbf{G}}$. For an equivalent interpretation of spectral zeros as poles of an inverse system we refer to [25].

Theorem 2.1

Given system $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with transfer function $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ (assuming invertible $\mathbf{D} + \mathbf{D}^*$), the spectral zeros of Σ are the poles of the associated Hamiltonian system $\Sigma_h(\mathbf{E}_h, \mathbf{A}_h, \mathbf{B}_h, \mathbf{C}_h, \Delta)$, where:

$$\mathbf{A}_h = \begin{pmatrix} \mathbf{A} - \mathbf{B}\Delta\mathbf{C} & -\mathbf{B}\Delta\mathbf{B}^* \\ \mathbf{C}^*\Delta\mathbf{C} & -\mathbf{A}^* + \mathbf{C}^*\Delta\mathbf{B}^* \end{pmatrix}, \quad \mathbf{E}_h = \begin{pmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^* \end{pmatrix} \quad (2.2)$$

$$\mathbf{B}_h = \begin{pmatrix} \mathbf{B} \\ -\mathbf{C}^* \end{pmatrix} \Delta, \quad \mathbf{C}_h = -\Delta \begin{pmatrix} \mathbf{C} & \mathbf{B}^* \end{pmatrix}, \quad \Delta = (\mathbf{D} + \mathbf{D}^*)^{-1}. \quad (2.3)$$

Proof 2.1 See Appendix A.1.

Thus, spectral zeros of Σ are the generalized eigenvalues $s_1, \dots, s_{2n}$ of the *Hamiltonian eigenvalue problem* (2.4):

$$\mathbf{A}_h \mathbf{R} = \mathbf{E}_h \mathbf{R} \Lambda \quad (2.4)$$

where $\Lambda = \text{diag}(s_1, \dots, s_{2n})$. In (2.4), we partition the Hamiltonian eigenvalues $\Lambda = \text{diag}(\Lambda_-, \Lambda_+, \Lambda_\infty)$ according to their location in the left, right half of the complex plane or at ∞ respectively (i.e., the spectral zeros of Σ are grouped into *stable*, *antistable** and infinite). The right eigenvectors $\mathbf{R}$ are also partitioned accordingly:

$$\underbrace{\mathbf{A}_h \begin{pmatrix} \mathbf{X}_- & \mathbf{X}_+ & \mathbf{X}_\infty \\ \mathbf{Y}_- & \mathbf{Y}_+ & \mathbf{Y}_\infty \end{pmatrix}}_{\mathbf{R}} = \mathbf{E}_h \underbrace{\begin{pmatrix} \mathbf{X}_- & \mathbf{X}_+ & \mathbf{X}_\infty \\ \mathbf{Y}_- & \mathbf{Y}_+ & \mathbf{Y}_\infty \end{pmatrix}}_{\mathbf{R}} \Lambda. \quad (2.5)$$

It was shown in [14] that a passive and stable reduced model $\hat{\Sigma}(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \mathbf{D})$ of

*s $\in \mathbb{C}$ is stable if $\text{Re}(s) < 0$ and antistable (or “unstable”) otherwise

order k is obtained by constructing the projecting matrices as follows:

$$\mathbf{V} = \mathbf{X}_{-[:,1:k]}, \quad \mathbf{W} = \mathbf{Y}_{-[:,1:k]} \quad (2.6)$$

where the k columns of $\mathbf{X}_-$ and $\mathbf{Y}_-$ are chosen such that the corresponding spectral zeros $s_1, \dots, s_k$ all belong to left half plane (with (2.6), stable spectral zeros are interpolated, but the result also holds for antistable spectral zero interpolation; the requirement is that interpolated spectral zeros all belong to the same half plane).

$$\mathbf{V} = [(s_1 \mathbf{E} - \mathbf{A})^{-1} \mathbf{B}, \dots, (s_k \mathbf{E} - \mathbf{A})^{-1} \mathbf{B}] \quad (2.7)$$

$$\mathbf{W} = [(-s_1^* \mathbf{E}^* - \mathbf{A}^*)^{-1} \mathbf{C}^*, \dots, (-s_k^* \mathbf{E}^* - \mathbf{A}^*)^{-1} \mathbf{C}^*] \quad (2.8)$$

Through this construction, (2.7) and (2.8) are satisfied [14], and the reduced $\hat{\Sigma}$ interpolates Σ at the chosen s_i and their mirror images $-s_i^*$, $i = 1, \dots, k$ [13], [3]. The projection matrices $\mathbf{V}$ and $\mathbf{W}$ insure that the reduced system satisfies the positive real lemma [3], [13], [14], thus passivity is preserved.

We emphasize that there is no restriction on the singularity or definiteness of $\mathbf{E}$ with SZM. Even when $\mathbf{E}$ is singular and some of the associated spectral zeros are at ∞ , in $\mathbf{V}$ and $\mathbf{W}$ only the eigenvectors corresponding to finite spectral zeros are picked. No additional work is needed to deflate the infinite modes of the system to enforce invertibility of $\mathbf{E}$ (in [17], deflation of infinite modes is described as part of the balanced truncation method). SZM guarantees passivity by construction, because the reduced model interpolates the original passive system at spectral zeros, and the structure of the system matrices is irrelevant.

2.2.2 Dominant spectral zero selection

Having determined how to build matrices $\mathbf{V}$ and $\mathbf{W}$ to preserve passivity, the natural question is which k out of the n stable spectral zeros of Σ to pick, so that the approximation error $\|\Sigma - \hat{\Sigma}\|$ is small.

Given that spectral zeros of Σ are the poles of $\mathbf{G}(s)$ defined by (2.1), we propose a selection criterion based on the *dominance* of each spectral zero, which introduces the effect of the *residues* of the transfer function $\mathbf{G}(s)$.

Using ideas from modal approximation [11], [3], we express the partial fraction expansion of $\mathbf{G}(s)$, in terms of poles s_i of $\Sigma_{\mathbf{G}}$ and associated residues R_j [26], resulting in:

$$\mathbf{G}(s) = \sum_{j=1}^{2n} \frac{R_j}{s - s_j}. \quad (2.9)$$

Spectral zeros s_j and residues R_j are found via the Hamiltonian eigenvalue problem $(\Lambda, \mathbf{R}, \mathbf{L}) = \text{eig}(\mathbf{A}_h, \mathbf{E}_h)$, where:

$$\mathbf{A}_h \mathbf{R} = \mathbf{E}_h \mathbf{R} \Lambda, \quad \mathbf{L}^* \mathbf{A}_h = \Lambda \mathbf{L}^* \mathbf{E}_h, \quad \text{and}$$

$$\Lambda = \text{diag}(s_1, \dots, s_n, -s_1^*, \dots, -s_n^*)$$

$$\mathbf{R} = [\mathbf{r}_1, \dots, \mathbf{r}_{2n}], \quad \mathbf{L} = [\mathbf{l}_1, \dots, \mathbf{l}_{2n}].$$

Λ contains the spectral zeros of Σ . The residues R_j are computed from the right and left eigenvectors using:

$$\gamma_j = \mathbf{C}_h \mathbf{r}_j (\mathbf{l}_j^* \mathbf{E}_h \mathbf{r}_j)^{-1}, \quad \beta_j = \mathbf{l}_j^* \mathbf{B}_h, \quad R_j = \gamma_j \beta_j \quad (2.10)$$

Definition 2.1 Dominant spectral zeros s_j of Σ are the spectral zeros with largest associated measure (2.11), where residues R_j are computed from (2.10) [11]:

$$\frac{\|R_j\|_2}{|Re(s_j)|}. \quad (2.11)$$

A *modal approximant* $\hat{\Sigma}_{\mathbf{G}}$ of dimension $2k$ for the Hamiltonian system $\Sigma_{\mathbf{G}}$, can be obtained by interpolating $\Sigma_{\mathbf{G}}$ at its $2k$, $k \ll n$ most dominant poles (i.e., at the k dominant spectral zeros of Σ , together with their mirror images), such that the reduced Hamiltonian system $\hat{\Sigma}_{\mathbf{G}}$ has transfer function: $\hat{\mathbf{G}}(s) = \sum_{j=1}^{2k} \frac{R_j}{s-s_j}$. Using this observation, a *passive* reduced $\hat{\Sigma}$ for Σ , is obtained by first picking the k *most dominant spectral zeros* from Λ in the same half plane. The corresponding columns of $\mathbf{R}$ are then used to build matrices $\mathbf{V}$ and $\mathbf{W}$ as described by (2.6). $\mathbf{V}$ and $\mathbf{W}$ finally project Σ to the reduced $\hat{\Sigma}$ according to (1.3). A summary of the reduction procedure is given in Appendix A.2.

Classifying spectral zeros according to large (2.11) is motivated by the fact that peaks of the frequency response of $\mathbf{G}(s)$ typically occur close to frequencies ω_j , where $s_j = \sigma_j + i\omega_j$ [11], [3] (see also Fig. 2.11). However, depending on the application, one could select spectral zeros that are dominant in a different sense. For example, spectral zeros with small imaginary parts or small in absolute value could be preferred (see for instance [27]). Different selection criteria will yield different approximations.

2.2.3 Algorithm for dominant spectral zero approximation (SADPA)

To make the dominant spectral zero reduction method suitable for large scale applications, we propose an iterative procedure (developed in [11] and [15, Chapter 3]) for

computing efficiently dominant spectral zeros and associated $\mathbf{V}$ and $\mathbf{W}$ projecting matrices.

As seen in (2.10), to determine the dominant spectral zeros, all residues, spectral zeros and associated eigenvectors should be available. This requires full computation of the Hamiltonian eigenvalue problem (2.4), which is numerically infeasible for large scale applications. An efficient alternative is to use an *iterative algorithm* to approximately compute only the desired k most dominant eigentriplets $(s_j, \mathbf{r}_j, \mathbf{l}_j)$, $j = 1, \dots, k$ of $(\mathbf{A}_h, \mathbf{E}_h)$ (a pair of complex conjugate eigenvalues (s_j, s_j^*) counts as one spectral zero).

Difficulties in computation arise from the distribution of dominant spectral zeros, which can be scattered anywhere in the complex plane. The scope of popular Arnoldi-type algorithms is limited for such applications. For these methods, convergence starts with well separated eigenmodes located at the extremes of the spectrum. However, extreme eigenmodes are not necessarily dominant according to measure (2.11). Furthermore, dominance criteria other than (2.11) could be relevant, depending on the application. Thus, a more flexible iterative algorithm for eigenvalue computation is needed, where the desired dominance criterion is predefined and convergence occurs accordingly. This can be achieved through the *Subspace Accelerated Dominant Pole Algorithm* (SADPA) proposed in [11], [15, Chapter 3]. For an overview of SADPA and implementation details adapted for efficient spectral zero computation, see Appendix A.3.

In this work, SADPA is used to compute dominant spectral zeros of the original system as dominant poles of the associated Hamiltonian system. The implementation of SADPA from [11], [15, Chapter 3] is adapted here to account for the 4-fold symmetry (1.6) of spectral zeros, ensuring convergence respects this structure (for an

outline see Appendix A.3).

With SADPA, we compute the $(2n \times k)$ dimensional dominant right and left eigenspaces $\hat{\mathbf{R}} = [\mathbf{r}_1, \dots, \mathbf{r}_k] \in \mathbb{C}^{2n \times k}$ and $\hat{\mathbf{L}} = [\mathbf{l}_1, \dots, \mathbf{l}_k] \in \mathbb{C}^{2n \times k}$ of $(\mathbf{A}_h, \mathbf{E}_h)$, associated with the k most dominant spectral zeros of Σ . Similarly to (2.6), projection matrices $\mathbf{V}$ and $\mathbf{W}$ are constructed by choosing the k columns of $\hat{\mathbf{R}}$, that correspond to the stable spectral zeros. Finally $\mathbf{V}$ and $\mathbf{W}$ from (2.12) reduce Σ according to:

$$\mathbf{V} = \hat{\mathbf{R}}_{[1:n, 1:k]}, \quad \mathbf{W} = \hat{\mathbf{R}}_{[n+1:2n, 1:k]} \quad (2.12)$$

$$\hat{\Sigma} = \left[\begin{array}{c|c} \mathbf{W}^* \mathbf{E} \mathbf{V}, \mathbf{W}^* \mathbf{A} \mathbf{V} & \mathbf{W}^* \mathbf{B} \\ \hline \mathbf{C} \mathbf{V} & \mathbf{D} \end{array} \right] \quad (2.13)$$

2.3 Hamiltonian system variants

As shown in Sect. 2.2.1, the passivity preserving projection for an original system Σ is obtained from an associated Hamiltonian system. We address possible formulations for the Hamiltonian system under consideration. Although these Hamiltonian variants are theoretically equivalent, numerical differences occur in implementation with SADPA, influencing convergence rate and computational cost. We also treat reduction for systems with $\mathbf{D} = \mathbf{0}$ in the original Σ , providing a solution to problem (P3) mentioned in Sect. 1.1.

2.3.1 Structured Hamiltonian

In Sect. 2.2.1, we showed how spectral zeros of Σ are determined as poles of the associated Hamiltonian system Σ_h (2.2), (2.3). In (2.2) however, sparsity for $\mathbf{A}_h$ may be lost in certain applications, particularly when $\mathbf{B}$ and $\mathbf{C}$ are full. This degrades performance of SADPA (especially in the solve step 3, see Appendix A.3, Algorithm

1). We avoid this by replacing $\Sigma_h(\mathbf{E}_h, \mathbf{A}_h, \mathbf{B}_h, \mathbf{C}_h, \Delta)$ from (2.2), (2.3), with the equivalent structured Hamiltonian system $\Sigma_s(\mathbf{E}_s, \mathbf{A}_s, \mathbf{B}_s, \mathbf{C}_s, \Delta)$ in the form (2.14), (A.7), which preserves sparsity. This is demonstrated through the following theorem.

Theorem 2.2

Given system $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with transfer function $\mathbf{H}(s)$, and the structured Hamiltonian system $\Sigma_s(\mathbf{E}_s, \mathbf{A}_s, \mathbf{B}_s, \mathbf{C}_s, \Delta)$, with:

$$\mathbf{A}_s = \begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & -\mathbf{A}^* & -\mathbf{C}^* \\ \mathbf{C} & \mathbf{B}^* & \mathbf{D} + \mathbf{D}^* \end{pmatrix}, \mathbf{E}_s = \begin{pmatrix} \mathbf{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \mathbf{B}_s = \begin{pmatrix} \mathbf{B} \\ -\mathbf{C}^* \\ \mathbf{0} \end{pmatrix} \Delta \quad (2.14)$$

$$\mathbf{C}_s = -\Delta \begin{pmatrix} \mathbf{C} & \mathbf{B}^* & \mathbf{0} \end{pmatrix}, \Delta = (\mathbf{D} + \mathbf{D}^*)^{-1} \quad (2.15)$$

the following holds (assuming invertible $\mathbf{D} + \mathbf{D}^*$):

- 1) $\Sigma_s(\mathbf{E}_s, \mathbf{A}_s, \mathbf{B}_s, \mathbf{C}_s, \Delta)$ has transfer function $\mathbf{G}(s) = [\mathbf{H}(s) + \mathbf{H}^*(-s)]^{-1}$, the same as the system $\Sigma_h(\mathbf{E}_h, \mathbf{A}_h, \mathbf{B}_h, \mathbf{C}_h, \Delta)$ specified by (2.2), (2.3).
- 2) Spectral zeros of Σ are the generalized eigenvalues of $(\mathbf{A}_s, \mathbf{E}_s)$ in the sparse form (2.14), the same as the generalized eigenvalues of the pair $(\mathbf{A}_h, \mathbf{E}_h)$ in the form (2.2).

Proof 2.2 See Appendix A.1.

The Hamiltonian eigenvalue problem (2.4) in structured form becomes $\mathbf{A}_s \mathbf{R} = \mathbf{E}_s \mathbf{R} \mathbf{A}$, and SADPA is applied on the structured Hamiltonian $\Sigma_s(\mathbf{E}_s, \mathbf{A}_s, \mathbf{B}_s, \mathbf{C}_s, \Delta)$, without affecting sparsity.

2.3.2 The $\mathbf{D}=0$ case

According to [13, 14], the theory in sections 2.2 and 2.3.1 assumed $\mathbf{D} \neq 0$ in the output equation of the original system, so that the reduced model (2.13) is strictly passive. Often, for systems arising in circuit simulation, the original differential equations are formulated so that $\mathbf{D} = 0^\dagger$. In this case, the construction of the Hamiltonian whose poles are the spectral zeros of $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ is shown in the following theorem.

Theorem 2.3

The structured Hamiltonian system associated with $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ has the form:

$$\begin{aligned} \mathbf{A}_s &= \begin{pmatrix} \mathbf{A} & 0 & \mathbf{B} \\ 0 & -\mathbf{A}^* & -\mathbf{C}^* \\ \mathbf{C} & \mathbf{B}^* & 0 \end{pmatrix}, \quad \mathbf{E}_s = \begin{pmatrix} \mathbf{E} & 0 & 0 \\ 0 & \mathbf{E}^* & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \mathbf{B}_s &= \begin{pmatrix} \mathbf{B} \\ -\mathbf{C}^* \\ \mathbf{I} \end{pmatrix}, \quad \mathbf{C}_s = -\begin{pmatrix} \mathbf{C} & \mathbf{B}^* & \mathbf{I} \end{pmatrix}. \end{aligned}$$

Proof 2.3 See Appendix A.1.

For such $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$, the projection (2.13) (with $\mathbf{W}$ and $\mathbf{V}$ computed either from the Hamiltonian eigenvectors (2.12), or via (2.7) and (2.8)), yields a lossless reduced system, with all poles on the $j\omega$ axis [28]. When $\mathbf{D} = 0$, the projection which guarantees a stable and strictly passive reduced model turns out to be a slight modification to (2.13), and is treated next.

[†]For the purpose of this chapter, we assume SISO systems, where $\mathbf{D}$ is scalar.

We evaluate the transfer function at $s = \infty$ according to (2.16), and distinguish two cases: $\mathbf{H}(s)|_{s=\infty} = 0$ and $\mathbf{H}(s)|_{s=\infty} \neq 0$.

$$\mathbf{H}(s)|_{s=\infty} = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}|_{s=\infty} = \tilde{\mathbf{D}} \quad (2.16)$$

Case $\mathbf{H}(\infty) = 0$

As mentioned above, reduction according to (2.13) with $\mathbf{D} = 0$ would yield a lossless system, with all poles on the $j\omega$ axis [28]. To guarantee reduced models with poles strictly in the left half plane, we solve the modified Hamiltonian eigenvalue problem (2.17), where $\delta \neq 0$ replaces $\mathbf{D} + \mathbf{D}^* = 0$ in (2.14). Lemma 2.1 summarizes the complete stability and passivity preserving reduction procedure.

$$\begin{pmatrix} \mathbf{A} & 0 & \mathbf{B} \\ 0 & -\mathbf{A}^* & -\mathbf{C}^* \\ \mathbf{C} & \mathbf{B}^* & \delta \end{pmatrix} \begin{pmatrix} \mathbf{X}_\delta \\ \mathbf{Y}_\delta \\ \mathbf{Z}_\delta \end{pmatrix} = \begin{pmatrix} \mathbf{E} & 0 & 0 \\ 0 & \mathbf{E}^* & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X}_\delta \\ \mathbf{Y}_\delta \\ \mathbf{Z}_\delta \end{pmatrix} \Lambda_\delta \quad (2.17)$$

Lemma 2.1

Given an original system with $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ and $\mathbf{H}(\infty) = 0$, a stable and passive reduced system is obtained as follows:

- 1) *Solve the modified Hamiltonian eigenvalue problem (2.17) with large δ ($\delta \in [10^4, 10^6]$ in practice), using SADPA.*
- 2) *Partition $\mathbf{X}_\delta$ and $\mathbf{Y}_\delta$ corresponding to the stable, anti-stable and infinite Hamiltonian eigenvalues Λ_δ . Choose the k most dominant spectral zeros from Λ_δ (in the left half plane) and build the corresponding projecting matrices $\tilde{\mathbf{X}}_\delta$, $\tilde{\mathbf{Y}}_\delta$,*

according to (2.18):

$$\begin{pmatrix} \mathbf{X}_\delta \\ \mathbf{Y}_\delta \\ \mathbf{Z}_\delta \end{pmatrix} = \begin{pmatrix} \mathbf{X}_{\delta,-} & \mathbf{X}_{\delta,+} & \mathbf{X}_{\delta,\infty} \\ \mathbf{Y}_{\delta,-} & \mathbf{Y}_{\delta,+} & \mathbf{Y}_{\delta,\infty} \\ \mathbf{Z}_{\delta,-} & \mathbf{Z}_{\delta,+} & \mathbf{Z}_{\delta,\infty} \end{pmatrix}, \quad \begin{aligned} \tilde{\mathbf{X}}_\delta &= \mathbf{X}_{\delta,-[1:k]} \\ \tilde{\mathbf{Y}}_\delta &= \mathbf{Y}_{\delta,-[1:k]} \\ \tilde{\mathbf{Z}}_\delta &= \mathbf{Z}_{\delta,-[1:k]} \end{aligned} \quad (2.18)$$

3) Obtain the reduced system by projection, keeping $\mathbf{D} = 0$:

$$\hat{\Sigma} = \left[\begin{array}{c|c} \tilde{\mathbf{Y}}_\delta^* \mathbf{E} \tilde{\mathbf{X}}_\delta, \tilde{\mathbf{Y}}_\delta^* \mathbf{A} \tilde{\mathbf{X}}_\delta & \tilde{\mathbf{Y}}_\delta^* \\ \hline \mathbf{C} \tilde{\mathbf{X}}_\delta & 0 \end{array} \right]. \quad (2.19)$$

$\hat{\Sigma}$ from (2.19) is stable and strictly passive, and interpolates $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ at points close to the dominant poles of $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$, together with their mirror images.

Proof 2.4 See Appendix A.1.

In particular, δ is chosen so that the poles of the projected system are shifted away from the $j\omega$ axis, resolving the above mentioned problem. The question arises of how to appropriately choose δ , such that the resulting projection also gives meaning to the approximation result. Inspired by the observation that dominant spectral zeros and dominant poles are closely related (see Sect. 2.4.4), it turns out that $\delta \rightarrow \infty$ transforms (2.17) into an eigenvalue problem whose solution Λ_δ are points close to the poles of the original system, *together with their mirror images*. Interpolation at these almost dominant poles and their mirror images gives a strictly passive reduced model that is implicitly stable and furthermore embeds characteristics of the system response, similar to modal approximation. Finally, by keeping $\mathbf{D} = 0$ in the reduced

model, the interpolating projectors $\tilde{\mathbf{X}}_\delta$ and $\tilde{\mathbf{Y}}_\delta$ reduce the original $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ so that the response $\mathbf{H}(s)|_{s=\infty}=0$ is also captured.

Case $\mathbf{H}(\infty) \neq 0$

In this case, the transfer function response at $s=\infty$ reveals a value for $\tilde{\mathbf{D}} \neq 0$, even if the original data is provided as $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$. This is because $\tilde{\mathbf{D}} \neq 0$ is actually embedded in $\mathbf{A}$, and can be recovered by evaluating (2.16). $\tilde{\mathbf{D}}$ is then used in constructing the stable, passive, reduced model, which interpolates the original at dominant spectral zeros. To this end, SADPA computes the most dominant spectral zeros and the associated right/left eigenvectors vectors from the Hamiltonian eigenvalue problem:

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & -\mathbf{A}^* & -\mathbf{C}^* \\ \mathbf{C} & \mathbf{B}^* & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} = \begin{pmatrix} \mathbf{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} \Lambda \quad (2.20)$$

Grouping the right eigenvectors corresponding to the stable, anti-stable and infinite spectral zeros, we construct the projecting matrices $\tilde{\mathbf{X}}$, $\tilde{\mathbf{Y}}$ and $\tilde{\mathbf{Z}}$ as in (2.18). The non-zero $\tilde{\mathbf{D}}$ computed from (2.16), together with $\tilde{\mathbf{X}}$, $\tilde{\mathbf{Y}}$ and $\tilde{\mathbf{Z}}$ reduce the original system according to:

$$\hat{\Sigma} = \left[\begin{array}{c|c} \tilde{\mathbf{Y}}^* \mathbf{E} \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}^* \mathbf{A} \tilde{\mathbf{X}} + \tilde{\mathbf{Z}}^* \tilde{\mathbf{D}} \tilde{\mathbf{Z}} & \tilde{\mathbf{Y}}^* \mathbf{B} - \tilde{\mathbf{Z}}^* \tilde{\mathbf{D}} \\ \hline \mathbf{C} \tilde{\mathbf{X}} - \tilde{\mathbf{D}} \tilde{\mathbf{Z}} & \tilde{\mathbf{D}} \end{array} \right]. \quad (2.21)$$

$\hat{\Sigma}$ from (2.21) is stable, passive, and interpolates the original $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ at the selected dominant spectral zeros from Λ in (2.20) [28]. Projection (2.21) extracts

$\mathbf{H}(\infty) = \tilde{\mathbf{D}} \neq 0$ from the structure of $\mathbf{A}$, ensuring the passivity preserving projection, and capturing the non-zero response $\mathbf{H}(s)|_{s=\infty} \neq 0$ in the reduced model.

2.4 Numerical results

Several examples from circuit simulation are presented, including transmission line models with voltage controlled current sources (VCCs). We show that dominant SZM is more general than existing techniques such as PRIMA [4], and achieves accurate passive reduction with no restrictions on the system matrices. Dominant SZM is compared with modal approximation, PRIMA and PRBT, in terms of approximation quality and performance. We also motivate the selection of dominant spectral zeros over non-dominant spectral zeros for interpolation, and identify close associations between dominant spectral zeros and dominant poles. All simulations are carried out in Matlab 7.1 on an Intel(R) Core(TM)2 Duo CPU T7700, 2.40GHz. Regarding the presentation of results: (1) in the frequency response plots, the x-axis is logarithmic, i.e 8 stands for frequency 10^8 , (2) in all tables, the relative approximation error $\frac{\|\mathbf{H}(j\omega) - \mathbf{H}_k(j\omega)\|_2}{\|\mathbf{H}(j\omega)\|_2}$ is measured, where frequency ω sweeps the plotted frequency range, (3) the PRBT implementation uses exact solutions of the positive real Riccati equations, which may be time consuming. For advanced implementations of PRBT, see [29], [30].

2.4.1 Reduction of transmission line model 1 with VCCs

For the transmission line model with voltage controlled current sources (VCCs) in Fig. 2.1, the underlying equations yield system matrices which violate (1.11), thus passive reduction via PRIMA is unfeasible. The system is stable and passive, and the circuit simulator generates $n = 149$ internal variables. The dip in the frequency

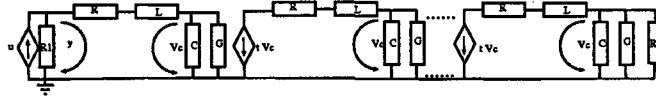


Figure 2.1 : Transmission line model, voltage controlled current sources unconnected to the previous block. $R_1=50\Omega$, $R_2=50\Omega$, $R=1\Omega$, $G=1pS$, $C=1pF$, $L=1\mu H$, $t=1$, $\|u\|=1/50A$.

response at $\omega = 10^9$ rad/s in Fig. 2.2, corresponds to one dominant spectral zero quadruple (together with a dominant pole pair) shown in Fig. 2.3. This suggests that interpolation at this dominant spectral zero quadruple should already give an accurate, stable and passive reduced order model of dimension $k = 2$. Indeed, dom-

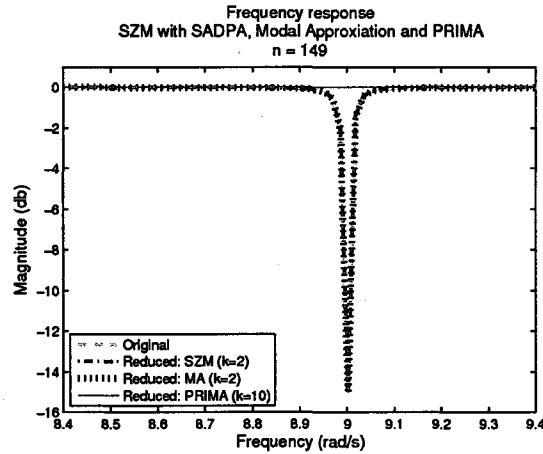


Figure 2.2 : VCC transmission line in Fig. 2.1. Original system and reduced with dominant SZM and modal approximation are indistinguishable.

inant SZM successfully reduces this stable, passive system, as seen from Fig. 2.2. The dip is accurately reproduced, because dominant SZM interpolates the original system at the dominant spectral zero. Modal approximation also yields excellent reduction, but SZM has the additional advantage of guaranteeing passivity. We attempted reduction via PRIMA (Fig. 2.2), and the reduced model is unstable and misses the dip in the frequency response. Table 2.1 summarizes the approximation

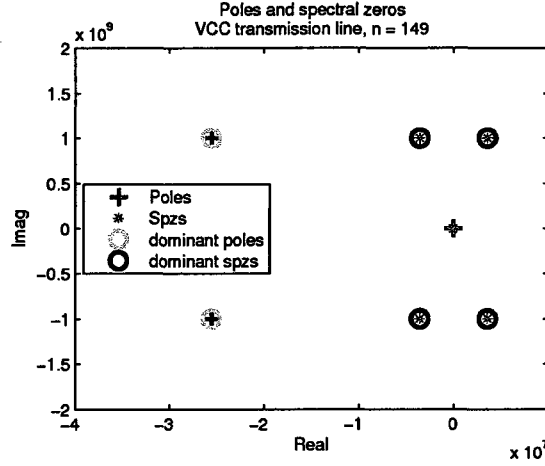


Figure 2.3 : VCC transmission line in Fig. 2.1. Dominant poles and spectral zeros (complex, encircled) are significantly larger than a cluster of non-dominant, real, poles and spectral zeros closer to the origin. Dominant poles and spectral zeros generate the dip at $\omega = 10^9$ rad/s in Fig. 2.2.

Table 2.1 : Approximation error for VCC transmission line in Fig. 2.1

Reduction: $n = 149, k = 2$	$\frac{\ H(j\omega) - H_k(j\omega)\ _2}{\ H(j\omega)\ _2}$
Dominant SZM	$1.75 \cdot 10^{-16}$
MA	$3.6 \cdot 10^{-16}$
PRIMA	unsuitable $A + A^*$ indefinite

results. An accurate, stable and passive reduced-order model is obtained automatically from $\Sigma(E, A, B, C, 0)$ even when the circuit equations cannot be formulated such that (1.10) and (1.11) are satisfied.

2.4.2 Reduction of transmission line model 2 with VCCs

We reduce the transmission line model with VCCs in Fig. 2.4. The main difficulty in reducing this system arises from the structure of the underlying system matrices, as E is singular and indefinite. PRBT [12] does not apply to E singular, and a deflation

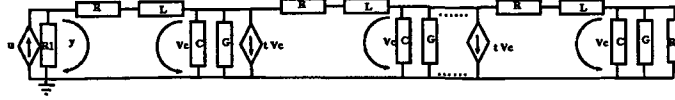


Figure 2.4 : Transmission line model with voltage controlled current sources connected to previous blocks. $R_1 = 50\Omega$, $R_2 = 50\Omega$, $R = 1\Omega$, $G = 1pS$, $C = 1pF$, $L = 1\mu H$, $t = 1$, $\|u\| = 1/50A$.

step is required to make $\mathbf{E}$ invertible. For large systems this becomes computationally infeasible. PRIMA [4] on the other hand requires a definite $\mathbf{E}$ and an $\mathbf{A}$ matrix with special block symmetries (1.11). To meet these assumptions, equations (1.1) should first be manipulated so that (1.10) is satisfied. This amounts to negating the stamps in $\mathbf{E}$ and $\mathbf{A}$ corresponding to the loop current variables, so that $\mathbf{E}$ becomes definite. Although not computationally expensive, applying the sign change transformation correctly requires knowing where the current variables are generated in the state vector. According to [21], this makes reduction non-automatic and cumbersome in practice. Dominant SZM is more general, avoiding both the deflation step required by PRBT and unlike PRIMA, is not confined to MNA-type models. Modal approximation [31], [32], [9], [10] also reduces the system with indefinite and singular $\mathbf{E}$ automatically, however dominant SZM has the additional feature of preserving passivity, and is shown here to yield a more accurate reduced model.

Fig. 2.5 shows the reduced models for dominant SZM, modal approximation and PRIMA. The system's transfer function is a simple high pass filter, but nevertheless clearly illustrates a case where dominant SZM achieves more accurate reduction than other methods. Already for $k = 2$, dominant SZM gives a passive reduced model which is indistinguishable from the original. Modal approximation only captures the dominant pole, but cannot reproduce the frequency response for the entire frequency range. The PRIMA model (stable only after conversion to $\mathbf{E}$ definite), matches the

Table 2.2 : Transmission line in Fig. 2.4, $n=1501$ $k=2$, reduction summary

SADPA based	Error	Time(s)	Iterations
Dominant SZM	$1.89 \cdot 10^{-4}$	0.8	14
MA	1.24	0.98	35
Other methods	Error	Time(s)	Constraints
PRIMA	$9.98 \cdot 10^{-1}$	0.01	E definite otherwise unstable
PRBT	-	-	E singular deflation unfeasible

response only in the lower frequency range, indicating that a reduced model of order $k > 2$ is needed for better approximation at higher frequencies (this was achieved for $k = 12$ - not shown). Approximation errors and CPU times are summarized in Table

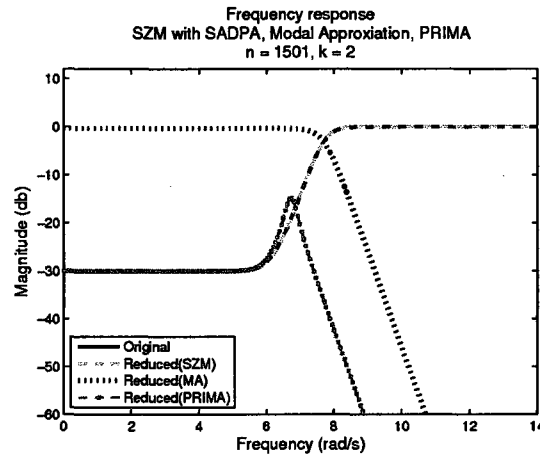


Figure 2.5 : VCC transmission line from Fig. 2.4. Frequency response of original system and reduced with dominant SZM, modal approximation and PRIMA.

2.2. Computing two dominant spectral zeros via SADPA enables efficient reduction ($t = 0.8s$) with dominant SZM, when reduction with PRBT [12] is computationally unfeasible. Passivity is preserved automatically even when **E** is indefinite.

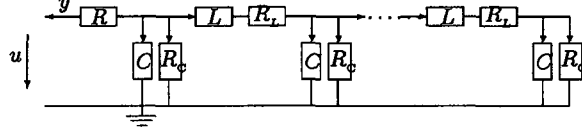


Figure 2.6 : RLC Transmission line. $R=10\Omega$, $R_C=10^8\Omega$, $C=10nF$, $L=1\mu H$, $R_L=0.001\Omega$, $\|u\|=1V$.

2.4.3 Reduction of RLC transmission line

We reduce the driving point admittance for the transmission line in Fig. 2.6. Unlike the previous transmission line models, this system exhibits many dominant spectral zeros and poles. These are computed efficiently with SADPA and dominant SZM captures the system's response at these points. Results are compared with reduction via modal approximation (MA), positive real balanced truncation (PRBT) [12] and PRIMA [4]. Finally, we motivate interpolation at dominant spectral zeros over spectral zeros that are less dominant: by analyzing spectrum plots for the reduced models, we identify a connection between the location of dominant spectral zeros and dominant poles in the complex plane.

The simulator generates $n = 902$ initial internal variables (dimension of original system), of which 599 are independent (dimension of deflated system, after the infinite modes have been eliminated). We reduce the system dimension to $k = 21$. Similarly to example 2.4.2, $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ has an underlying singular and indefinite $\mathbf{E}$.

The slow decay rate of the system positive real Hankel singular values, shown in Fig. 2.7, indicates a priori that the system is difficult to approximate, especially with PRBT (for a definition of positive real Hankel singular values and their relation to PRBT see [12], [3]).

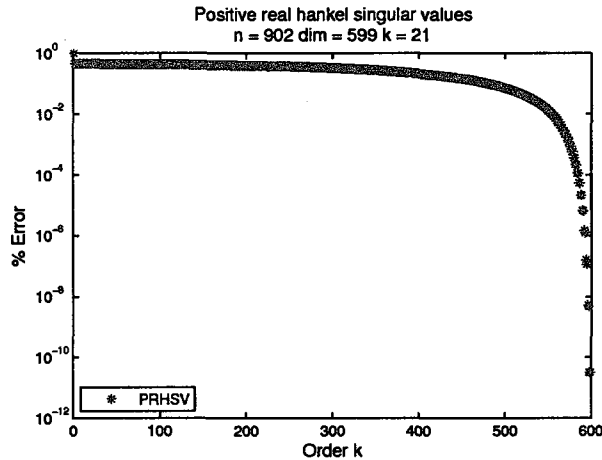


Figure 2.7 : RLC transmission line in Fig. 2.6. Normalized positive real Hankel singular values.

Dominant SZM and modal approximation, using SADPA

Fig. 2.8 shows the frequency response of the original and reduced system with dominant SZM and modal approximation, where the dominant spectral zeros and respectively dominant poles were computed with SADPA. The response is highly oscillatory and thus difficult to approximate (as already predicted from Fig. 2.7). Dominant SZM captures this behavior well for low and high frequencies, by interpolating at the dominant spectral zeros. This is contrasted by a surprisingly poor reduced model obtained with modal approximation. Interpolating the original system at dominant poles seems insufficient for good approximation.

Positive real balanced truncation

PRBT [12] was applied after deflating the infinite modes, so that $\mathbf{E}$ becomes invertible (recall that deflation was not necessary with dominant SZM). Comparing Figs. 2.9 and 2.8, dominant SZM captures the system's oscillatory response better than PRBT.

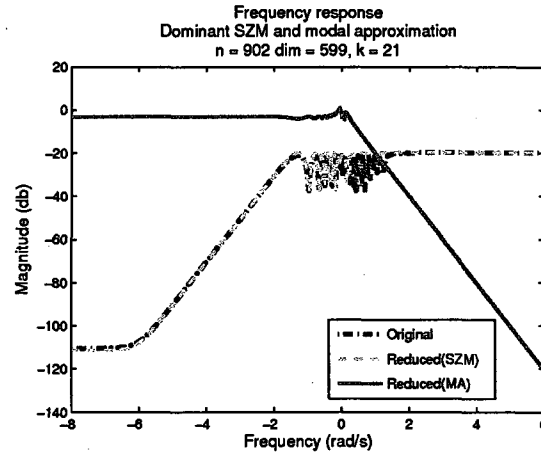


Figure 2.8 : RLC transmission line in Fig. 2.6. Frequency response of original system and reduced with dominant SZM and modal approximation.

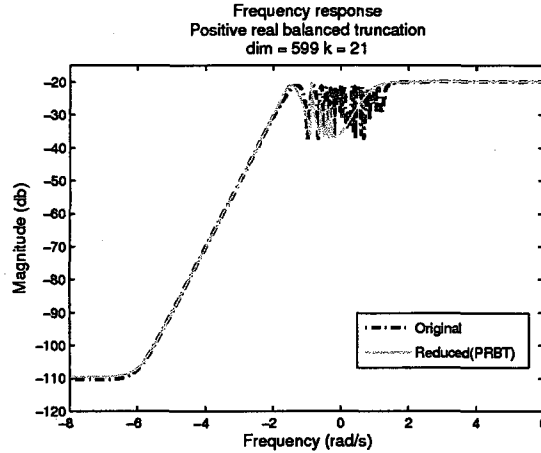


Figure 2.9 : RLC transmission line in Fig. 2.6. Frequency response of original system and reduced with positive real balanced truncation.

PRIMA

The performance of the industry standard reduction method, PRIMA [4], is shown in Fig. 2.10. As in [21], when PRIMA is applied directly on $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with $\mathbf{E}$ indefinite the reduced system is unstable. To guarantee stability and passivity, reduction with PRIMA becomes non-automatic - for (1.10) to hold, stamps in $\mathbf{E}$ and $\mathbf{A}$ corresponding to loop currents have to be negated. The matching of moments at

$s = 0$ for PRIMA is reflected in good approximation for low frequencies, but misses the response at higher frequencies.

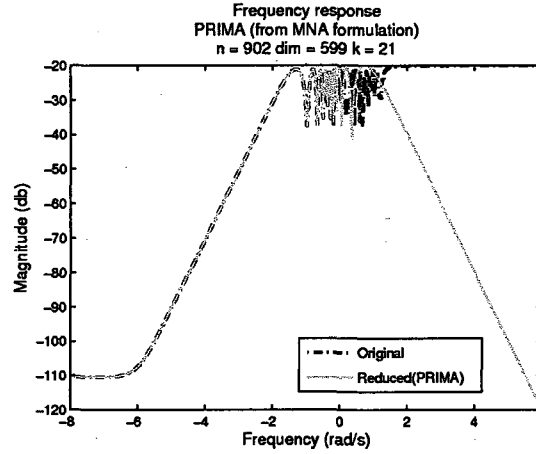


Figure 2.10 : RLC transmission line in Fig. 2.6. Frequency response of original system and reduced with PRIMA.

Table 2.3 summarizes approximation error and computational cost for all methods applied. Dominant SZM and PRBT give similar approximation errors, however PRBT requires a costly deflation step of infinite modes, which is impractical. Not counting the cost of deflation, CPU time for PRBT was $\simeq 272$ s (implemented with the MATLAB `care` routine), significantly higher than for dominant SZM ($\simeq 5.36$ s). More efficient implementations of PRBT for large systems with $\mathbf{E}$ singular go beyond the scope of this thesis, see [29, 30] for possible approaches. Even though PRIMA is the cheapest method (0.21 s), it gives poor approximation at high frequencies. In contrast with PRIMA, dominant SZM reduces the system with indefinite $\mathbf{E}$ directly and the resulting reduced model is more accurate. For this example, dominant SZM provides the best trade-off between approximation quality, preservation of stability and passivity and applicability on the system with singular and indefinite $\mathbf{E}$ directly, without intervening in the system structure a priori.

Table 2.3 : RLC transmission line in Fig. 2.6, $n = 902$, $k = 21$, reduction summary for various methods

SADPA based	Error	Time(s)	Iterations
Dominant SZM	0.2251	5.36	105
MA	8.17	5.24	205
Other methods	Error	Time(s)	<i>Constraints</i>
PRBT	0.233	272	E non-singular deflation $t=896$ s
PRIMA	0.876	0.21	E definite otherwise unstable

2.4.4 Interpolation at dominant versus non-dominant spectral zeros. A connection to dominant poles.

We motivate the selection of dominant spectral zeros for interpolation, where the dominance criterion is (2.11), and reveal a connection between dominant spectral zeros and dominant poles.

Since spectral zeros of the original system $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ are poles of the associated Hamiltonian system $\Sigma_s(\mathbf{E}_s, \mathbf{A}_s, \mathbf{B}_s, \mathbf{C}_s, \Delta)$ specified by (2.14), (A.7), we inspect the frequency response of the Hamiltonian system and its modal approximant (shown in Fig. 2.11). Peaks in the frequency response of the Hamiltonian system occur at frequencies close to the imaginary parts of its dominant poles. Modal approximation on the Hamiltonian system $\Sigma_s(\mathbf{E}_s, \mathbf{A}_s, \mathbf{B}_s, \mathbf{C}_s, \Delta)$ preserves these dominant poles and implicitly the dominant spectral zeros of the original system. This gives intuition for reducing the original system $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ directly with the dominant SZM. The result for dominant SZM was already shown in Fig. 2.8. Shown in Fig. 2.12 is the spectrum of spectral zeros and poles for the original and reduced system with dominant SZM. The reduced system interpolates the original at the dominant spectral zeros computed with SADPA (green circles match red stars). The location of

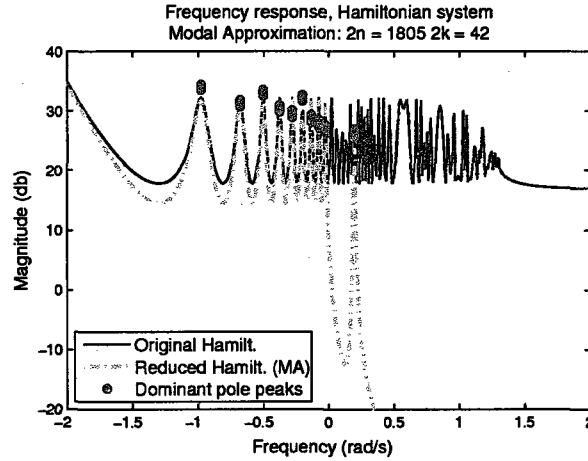


Figure 2.11 : RLC transmission line in Fig. 2.6. Frequency response of original Hamiltonian system and reduced Hamiltonian system with modal approximation (dominant poles of Hamiltonian system are dominant spectral zeros of original system).

poles for the reduced model with dominant SZM (light-blue squares) gives additional insight into the method's features: interpolation at dominant spectral zeros places poles for the reduced model in patterns similar to the original dominant poles (orange triangles). To further support the connection between dominant spectral zeros,

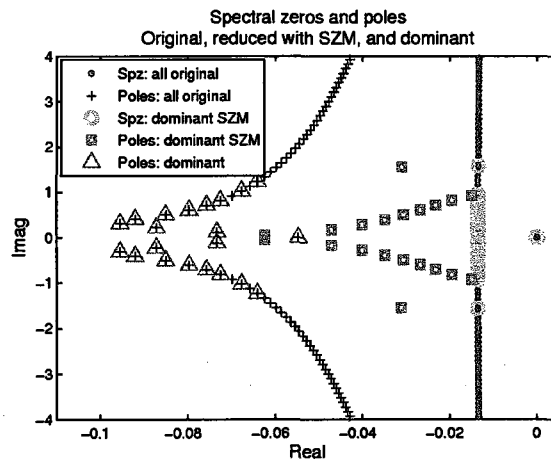


Figure 2.12 : RLC transmission line in Fig. 2.6. Stable spectral zeros and poles of original system and reduced with dominant SZM. Interpolation is at dominant spectral zeros (green circles). Reduced order poles (light-blue squares) follow similar patterns to original dominant poles (orange triangles).

dominant poles and approximation quality, Fig. 2.13 shows poor SZM approximation via interpolation at *non-dominant* spectral zeros (for instance spectral zeros close to the imaginary axis). The reduced model captures no oscillations from the frequency response of the original system, contrasting the result in Fig. 2.8 when dominant spectral zeros were interpolated. From the spectrum in Fig. 2.14, we see that interpolation at non-dominant spectral zeros (green circles) also generates poles at non-dominant locations (light-blue squares), far from the location of original dominant poles (orange triangles in Fig. 2.12).

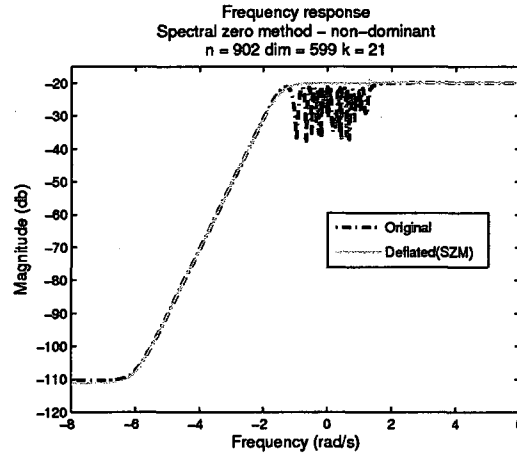


Figure 2.13 : RLC transmission line in Fig. 2.6. Frequency response of original system and reduced with SZM, with interpolation at non-dominant spectral zeros. Oscillations are missed.

Summarizing the observations above, dominant SZM combines two key ingredients to compute passive reduced models: interpolation at dominant points and passivity preservation by construction. As with modal approximation (which preserves dominant poles [31, 32, 9, 10]), a reduced model is constructed via interpolation at points that characterize the system’s behavior. Dominant SZM chooses these points as the dominant spectral zeros of the original system, guaranteeing a passive reduced model [13], [14]. Our experiments show in addition that dominant spectral zeros are associ-

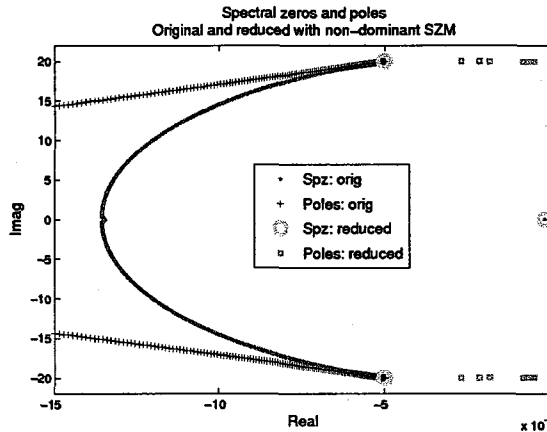


Figure 2.14 : RLC transmission line in Fig. 2.6. Stable spectral zeros and poles of original system and reduced with non-dominant SZM. Interpolation is at spectral zeros close to the imaginary axis (green circles). Reduced order poles (light-blue squares) are far from the location of original dominant poles.

ated with poles whose location in the complex plane resemble dominant poles. This observation also supports the passivity preserving projection for the case $\mathbf{D} = 0$ in Sect. 2.3.2, 1). There, the spectral zero interpolation problem is modified into an approximate modal approximation problem, where interpolation is at points close to the dominant poles of the original system, together with their mirror images. We note that such an approach could also be exploited for the case $\mathbf{D} \neq 0$, but is unnecessary for the purpose of this work and will be addressed in the future.

2.5 Complexity, convergence and error control

We address complexity and convergence for the dominant spectral zero reduction method. Our discussion[†] follows the implementation of dominant SZM via the iterative dominant eigenvalue solver, SADPA [11], [15, Chapter 3]. Appendix A.3 shows

[†]Section due to Joost Rommes at NXP Semiconductors, as part of [16].

an algorithmic pseudocode for SADPA, which has been adapted here efficiently for computing dominant spectral zeros.

2.5.1 Complexity and comparison

The main computational costs in dominant SZM are in steps 3 and 4 of Alg. 1: theoretically speaking, the costs for solving a single linear system, such as in step 3, are $O(n^3)$, which would make the algorithm impractical for large systems. In practice, however, circuit system matrices $\mathbf{A}$ and $\mathbf{E}$ are sparse, and by making use of clever row/column reorderings, fill-in of the LU factors can be minimized. Hence the linear systems in step 3 and 4 can be solved much more efficiently. Note that a single LU factorization is needed per iteration, since the factorization U^*L^* can be used in step 4. Our implementation used the backslash operator in Matlab, which turned out to be very efficient. Practical experience shows factorization and solution costs of $O(n^\alpha)$ with $1 < \alpha < 2$ for typical circuit matrices, see also [33].

The Modified Gram-Schmidt procedure in steps 5 and 6 of Alg. 1 is of $O(kn)$ per iteration and is the second largest contribution in general. By keeping the search spaces small these costs can be limited; a practical choice is to restart at $k_{max} \leq 15$ with a search space of dimension $k_{min} \geq 4$. For large search space dimensions, the costs for the eigendecomposition in step 1 of Alg. 2 can become large: for a search space of dimension k , the costs are $O(k^3)$, which indicates that $k_{max} \ll \sqrt{n}$ to avoid too high costs. All other operations are cheap compared to the described costs (assuming sparse matrices). Note that direct computation of the projection matrices via (2.7) and (2.8) is not needed since bases for the corresponding subspaces are available implicitly, as described in section 2.2.1, equations (2.5), (2.6).

In summary, the costs for the complete procedure are $O(q(k_{max}n + k_{max}^3 + 2n^\alpha))$,

where q is the number of iterations, $1 < \alpha < 2$, and $k_{max} \leq 15$. Compared to PRIMA, which has costs of $O(q^2n + qn^\alpha)$, the big difference is that our approach needs factorizations of $s_i\mathbf{E} - \mathbf{A}$ for multiple values of s_i , while PRIMA in principle only needs one factorization. Consequently, computational costs for PRIMA will be lower in general practice, provided that: (1) the order of the reduced order model should not be too big, otherwise the orthogonalization costs will start to dominate, and (2), only $O(1)$ interpolation points s_i are needed to produce an accurate model. Especially if condition (2) is not satisfied, costs for PRIMA may increase rapidly: first, one has to choose/compute the correct shift to improve accuracy, and then the Krylov bases must be computed. The dominant spectral zero approach, on the other hand, automatically finds the most effective shifts. Similar remarks can be made with respect to Poor Man's TBR[§] [33], with complexity $O(q^2n + 2qn^\alpha)$: although for PMTBR the number of frequency points q is in general lower than the number of iterations needed for our approach, one still has to rely on a frequency sample selection strategy [33]. However, in agreement with arguments in [33], the reduced model generation time is in practice dominated by the model simulation time, making model compactness, accuracy, and passivity more important than the reduction time. In this respect, the dominant spectral zero approach has the advantage over other methods that it is automatic and that it preserves passivity without putting constraints on the original system matrices or projections.

In case that exact solutions of the linear systems are not computable due to memory and CPU limits, one could use inexact solution methods (see [15, Chapter 3]) to find spectral zeros: only steps 3 and 4 of Alg. 1 need to be replaced by inexact

[§]An implementation of PMTBR is omitted here, as it is neither stability nor passivity preserving by default.

solutions methods. This will slow down convergence, but the reduction procedure remains the same and applicable in practice. In the case of PRIMA and PMTBR, for instance, such an adaptation is less trivial, since the availability of exact solutions for linear systems with operator $(s_0\mathbf{E} - \mathbf{A})$ is assumed.

2.5.2 Convergence and error control

Concerning the convergence to spectral zeros it is clear that, being a (subspace accelerated) Newton method, the rate of convergence is quadratic in the neighborhood of a spectral zero. Moreover, convergence to *dominant* spectral zeros is much more likely than convergence to less dominant zeros, even if located close to each other, due to the specific convergence properties of the Dominant Pole Algorithm. We refer the reader to [34] for a detailed analysis.

For the convergence of reduced order models to accurate approximations of the original systems, the situation is less trivial. Unlike balanced truncation, there is no computable error bound, as is the case for all Krylov-type methods, including PRIMA. A practical way to deal with this is to monitor the decay of the residues associated to the spectral zeros: if the residues of the most recent dominant zeros are significantly smaller than the residues of already found zeros, convergence could be reached. An additional check can be to inspect the difference $\frac{\|\mathbf{H}(j\omega_i) - \mathbf{H}_k(j\omega_i)\|}{\|\mathbf{H}(j\omega_i)\|}$ for a number of frequency samples ω_i .

2.6 Conclusions

Dominant SZM was presented, a passivity preserving reduction method which interpolates the dominant spectral zeros of the original system. The proposed method guarantees passivity for systems with more general structures than what can be han-

dled by industry-preferred techniques such as PRIMA [4]. Theoretical considerations for the dominant spectral zero method were outlined, and performance was evaluated by reducing the discretized models of transmission lines, including models with controlled sources (VCCs). Results showed that, irrespective of stability and passivity preservation, the performance of dominant SZM is at least comparable to passivity preserving methods like PRIMA and positive real balanced truncation, and even superior for certain examples. Dominant SZM is also suitable for reducing large scale systems, when implemented as an iterative eigenvalue approximation problem.

The spectral zero interpolation method SZM, first developed by [13] and [14], was extended here threefold. A classification of spectral zeros was proposed, based on a dominance criterion involving the associated residues. The second novel ingredient is the implementation using the recently developed subspace accelerated dominant pole algorithm (SADPA) [11], [15], which iteratively computes dominant spectral zeros irrespective of their location in the complex plane. Finally, based on the observation that dominant spectral zeros have close associations to dominant poles, we showed how for systems with $\mathbf{D}=0$, a projection is constructed which guarantees a strictly passive reduced model.

The ability of SADPA to approximate spectral zeros according to various dominance criteria makes SZM flexible in choosing the interpolated spectral zeros. Dominance criteria other than the proposed (2.11), that would yield significantly better approximants, are not known to the authors at this stage. A dominance criterion exploring $\mathcal{H}_2$ optimality [35] is currently under investigation. The MIMO implementation of dominant SZM, based on the multivariable version of SADPA [19], is treated in Sect. 4.4.

Chapter 3

A combined dominant SZM - Markov parameter matching approach for descriptor systems in circuit simulation

It is shown how modal approximation and the spectral zero interpolation method can be combined efficiently with moment matching at $s_0 = \infty$ for descriptor systems arising in circuit simulation, where the underlying system matrices satisfy certain structural properties. The approach is an alternative for resolving the lossless problem occurring with dominant SZM reduction when $\mathbf{D} = 0$.

3.1 Background

The combined framework proposed in this chapter can be placed in the context of partial realization of descriptor systems with singular $\mathbf{E}$ [36]. When $\mathbf{E}$ is singular, matching of moments at ∞ (also called Markov parameters) is an open problem, especially when no block structures of $\mathbf{A}$ and $\mathbf{E}$ are available. Similarly to [36], the approach uses information about the structure of $\mathbf{E}$, but requires little numerical work since only one or two Markov parameters are matched, instead of a full partial realization solution. The effort of the method proposed in this thesis is in computing the dominant spectral zero projection matrices (see Chapter 2), while the matching of few Markov parameters is integrated at the end. Two equivalent forms associated with a descriptor system $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ are addressed separately, and examples with both formulations are provided. The approach developed here for a combined

rational Krylov - partial realization framework continues the work in [36].

3.2 Equivalent form 1: slow-fast subsystem decomposition

Consider a descriptor system in the form (1.1), where without loss of generality assume $\mathbf{D} = 0$. According to [37], there exist nonsingular matrices $\mathbf{Q}$ and $\mathbf{P}$ such that (1.1) is *restricted system equivalent (r.s.e)* to the subsystem formulation (3.1), (3.2):

$$\dot{\mathbf{x}}_1(t) = \mathbf{A}_1 \mathbf{x}_1(t) + \mathbf{B}_1 \mathbf{u}(t) \quad (3.1)$$

$$\mathbf{y}_1(t) = \mathbf{C}_1 \mathbf{x}_1(t)$$

$$\mathbf{N} \dot{\mathbf{x}}_2(t) = \mathbf{x}_2(t) + \mathbf{B}_2 \mathbf{u}(t) \quad (3.2)$$

$$\mathbf{y}_2(t) = \mathbf{C}_2 \mathbf{x}_2(t)$$

$$\mathbf{y}(t) = \mathbf{C}_1 \mathbf{x}_1(t) + \mathbf{C}_2 \mathbf{x}_2(t) = \mathbf{y}_1(t) + \mathbf{y}_2(t).$$

The coordinate transformation is $[\mathbf{x}_1^T, \mathbf{x}_2^T]^T = \mathbf{P}^{-1} \mathbf{x}$, $\mathbf{x}_1 \in \mathbb{R}^{n_1 \times n_2}$, $\mathbf{x}_2 \in \mathbb{R}^{n_2 \times n_2}$ and the transformed system matrices are in the form (3.3):

$$\mathbf{QEP} = \begin{pmatrix} \mathbf{I}_{n_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}, \mathbf{QAP} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n_2} \end{pmatrix}, \mathbf{QB} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix}, (\mathbf{CP})^T = \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix}, \quad (3.3)$$

where $n_1 + n_2 = n$ and $\mathbf{N} \in \mathbb{R}^{n_2 \times n_2}$ is nilpotent with index ν , i.e. $\mathbf{N}^\nu = \mathbf{0}$. System (3.1) is called the *slow subsystem* and (3.2) the *fast subsystem*. R.s.e systems have identical transfer functions, thus $\mathbf{H}(s)$ can be split into (3.4), where $\mathbf{H}_1(s)$ is the strictly proper part and $\mathbf{H}_2(s)$ is the polynomial part of the transfer function.

$$\mathbf{H}(s) = \mathbf{C}_1(s\mathbf{I}_{n_1} - \mathbf{A}_1)^{-1}\mathbf{B}_1 + \mathbf{C}_2(s\mathbf{N} - \mathbf{I}_{n_2})^{-1}\mathbf{B}_2 = \mathbf{H}_1(s) + \mathbf{H}_2(s) \quad (3.4)$$

The Laurent series expansion of $\mathbf{H}(s)$ around infinity (3.5) reveals the Markov parameters of the slow and fast subsystems respectively [17],[37]:

$$\mathbf{H}_1(s) = \sum_{j=1}^{\infty} \underbrace{\mathbf{C}_1 \mathbf{A}_1^j \mathbf{B}_1}_{\eta_j} s^{-j}, \quad \mathbf{H}_2(s) = \sum_{j=0}^{\nu-1} \underbrace{-\mathbf{C}_2 \mathbf{N}^j \mathbf{B}_2}_{\mu_j} s^j. \quad (3.5)$$

In most circuit simulation applications, the transfer function is either proper or strictly proper. If proper, the polynomial part is a constant $\mathbf{H}_2(s) = -\mathbf{C}_2 \mathbf{B}_2 = \mu_0 \neq 0$ ($\mathbf{N}$ has index $\nu = 1$). Also $\mathbf{H}_2(s) = -\mathbf{C}_2 \mathbf{B}_2 = \lim_{s \rightarrow \infty} \mathbf{H}(s)$ is precisely the hidden $\tilde{\mathbf{D}} \neq 0$ in the context of Sect. 2.3.2. When $\mathbf{H}(s)$ is strictly proper $\mathbf{H}_2(s) = -\mathbf{C}_2 \mathbf{B}_2 = \mu_0 = 0$. Example 3.1 illustrates the transformation of a simple circuit to the equivalent form (3.1), (3.2).

Next, we introduce the notion of *dominant poles at ∞* for descriptor systems $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ with $\mathbf{E}$ singular, and show how interpolation at such poles is linked to moment matching at ∞ for the fast subsystem associated with $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$. We propose how to efficiently integrate interpolation at dominant poles at ∞ in the modal approximation (Sect. 3.2.1) and dominant SZM (Sect. 3.2.2) framework, when $\mathbf{B}$ and $\mathbf{C}^*$ belong to the nullspace of $\mathbf{E}$.

3.2.1 Modal approximation and poles at ∞

Consider descriptor systems $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ where without loss of generality, assume $\mathbf{D} = 0$ and $\mathbf{E}$ singular. Let n_1 and n_2 denote the number of finite and infinite eigenvalues of the pair $(\mathbf{A}, \mathbf{E})$ respectively, $n = n_1 + n_2$. Within the modal approximation framework, consider the partial fraction expansion [34] of the transfer function (3.6),

where R_∞ represents the constant contribution of the n_2 poles at ∞ :

$$\mathbf{H}(s) = \sum_{j=1}^{n_1} \frac{R_j}{s - \lambda_j} + R_\infty. \quad (3.6)$$

Lemma 3.1

The first Markov parameter of the fast subsystem associated with $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$, namely $\mu_0 = -\mathbf{C}_2 \mathbf{B}_2$ in (3.5), is the same as R_∞ , the total contribution of the poles at ∞ in (3.6).

Proof 3.1 Let $\mathbf{X} = [\mathbf{X}_{n_1}, \mathbf{X}_\infty]$ where $\mathbf{X}_{n_1} = [\mathbf{x}_1, \dots, \mathbf{x}_{n_1}] \in \mathbb{C}^{n \times n_1}$ spans the subspace of the n_1 right eigenvectors of $(\mathbf{A}, \mathbf{E})$ corresponding to the finite eigenvalues. $\mathbf{X}_\infty = [\mathbf{x}_{\infty_1}, \dots, \mathbf{x}_{\infty_{n_2}}] \in \mathbb{C}^{n \times n_2}$ spans the subspace of right eigenvectors of $(\mathbf{A}, \mathbf{E})$ corresponding to the n_2 eigenvalues at ∞ . The left eigenvectors of $(\mathbf{A}, \mathbf{E})$ are partitioned similarly, $\mathbf{Y} = [\mathbf{Y}_{n_1}, \mathbf{Y}_\infty]$, $\mathbf{Y}_{n_1} = [\mathbf{y}_1, \dots, \mathbf{y}_{n_1}]$, $\mathbf{Y}_\infty = [\mathbf{y}_{\infty_1}, \dots, \mathbf{y}_{\infty_{n_2}}]$.

For an eigentriplet $(\lambda_j, \mathbf{x}_j, \mathbf{y}_j)$ of $(\mathbf{A}, \mathbf{E})$, where λ_j is either a finite or infinite eigenvalue, the following holds [34]:

$$\mathbf{A} \mathbf{x}_j = \lambda_j \mathbf{E} \mathbf{x}_j, \quad \mathbf{x}_j \neq 0 \quad (3.7)$$

$$\mathbf{y}_j^* \mathbf{A} = \lambda_j \mathbf{y}_j^* \mathbf{E}, \quad \mathbf{y}_j \neq 0 \quad (3.8)$$

$$\mathbf{y}_i^* \mathbf{E} \mathbf{x}_j = 0, \quad \forall \lambda_i \neq \lambda_j, \quad i, j \leq n_1 \quad (3.9)$$

$$\mathbf{E} \mathbf{x}_{\infty_j} = 0, \quad \mathbf{y}_{\infty_j}^* \mathbf{E} = 0 \quad \forall \lambda_j = \infty, \quad j \leq n_2 \quad (3.10)$$

Assuming that eigenvectors corresponding to the finite eigenvalues are scaled so

that $\mathbf{y}_j^* \mathbf{E} \mathbf{x}_j = 1$, $i \leq n_1$, from (3.7)-(3.10) we obtain the projected system:

$$\begin{aligned}
 \mathbf{Y}^* \mathbf{E} \mathbf{X} &= \begin{pmatrix} \mathbf{Y}_{n_1}^* \mathbf{E} \mathbf{X}_{n_1} & \mathbf{Y}_{n_1}^* \mathbf{E} \mathbf{X}_\infty \\ \mathbf{Y}_\infty^* \mathbf{E} \mathbf{X}_{n_1} & \mathbf{Y}_\infty^* \mathbf{E} \mathbf{X}_\infty \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{n_1} & 0 \\ 0 & 0 \end{pmatrix}, \\
 \mathbf{Y}^* \mathbf{A} \mathbf{X} &= \begin{pmatrix} \mathbf{Y}_{n_1}^* \mathbf{A} \mathbf{X}_{n_1} & \mathbf{Y}_{n_1}^* \mathbf{A} \mathbf{X}_\infty \\ \mathbf{Y}_\infty^* \mathbf{A} \mathbf{X}_{n_1} & \mathbf{Y}_\infty^* \mathbf{A} \mathbf{X}_\infty \end{pmatrix} = \begin{pmatrix} \Lambda_{n_1} & \Lambda_{n_1} \mathbf{Y}_{n_1}^* \mathbf{E} \mathbf{X}_\infty \\ \Lambda_{n_1} \mathbf{Y}_\infty^* \mathbf{E} \mathbf{X}_{n_1} & \mathbf{Y}_\infty^* \mathbf{A} \mathbf{X}_\infty \end{pmatrix} \quad (3.11) \\
 &= \begin{pmatrix} \Lambda_{n_1} & 0 \\ 0 & \mathbf{Y}_\infty^* \mathbf{A} \mathbf{X}_\infty \end{pmatrix}, \quad \mathbf{Y}^* \mathbf{B} = \begin{pmatrix} \mathbf{Y}_{n_1}^* \mathbf{B} \\ \mathbf{Y}_\infty^* \mathbf{B} \end{pmatrix}, \quad \mathbf{C} \mathbf{X} = (\mathbf{C} \mathbf{X}_{n_1}, \mathbf{C} \mathbf{X}_\infty)
 \end{aligned}$$

Recall the existence of nonsingular matrices $\mathbf{Q}$ and $\mathbf{P}$ that put $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ in the form (3.3). Clearly, with $\mathbf{Q} = \mathbf{Y}^*$ and $\mathbf{P} = \mathbf{X}$, (3.11) is in the form (3.3) where:

$$\begin{aligned}
 \mathbf{A}_1 &= \Lambda_{n_1} = \text{diag}(\lambda_1, \dots, \lambda_{n_1}), \quad \mathbf{B}_1 = \mathbf{Y}_{n_1}^* \mathbf{B}, \quad \mathbf{C}_1 = \mathbf{C} \mathbf{X}_{n_1} \\
 \mathbf{N} &= 0 \in \mathbb{C}^{n_2 \times n_2}, \quad \mathbf{B}_2 = (\mathbf{Y}_\infty^* \mathbf{A} \mathbf{X}_\infty)^{-1} (\mathbf{Y}_\infty^* \mathbf{B}), \quad \mathbf{C}_2 = \mathbf{C} \mathbf{X}_\infty. \quad (3.12)
 \end{aligned}$$

The projected system is now decoupled into the slow and fast subsystems (3.1), (3.2). In particular, as $\mathbf{N} = 0$, for the fast subsystem we have: $0 = \mathbf{x}_2(t) + \mathbf{B}_2 \mathbf{u}(t)$, $\Rightarrow \mathbf{y}_2(t) = -\mathbf{C}_2 \mathbf{B}_2 \mathbf{u}(t)$ so the total system equations are rewritten in the form:

$$\dot{\mathbf{x}}_1(t) = \Lambda_{n_1} \mathbf{x}_1(t) + \mathbf{B}_1 \mathbf{u}(t) \quad (3.13)$$

$$\mathbf{y}(t) = \mathbf{C}_1 \mathbf{x}_1(t) - \mathbf{C}_2 \mathbf{B}_2 \mathbf{u}(t). \quad (3.14)$$

The transfer function becomes $\mathbf{H}(s) = \mathbf{C}_1 (s \mathbf{I}_{n_1} - \Lambda_{n_1})^{-1} \mathbf{B}_1 - \mathbf{C}_2 \mathbf{B}_2$. As in (3.4), the

strictly proper part corresponding to the slow subsystem is revealed:

$$\mathbf{H}_1(s) = \mathbf{C}_1(s\mathbf{I}_{n_1} - \mathbf{\Lambda}_{n_1})^{-1}\mathbf{B}_1 = \sum_{j=1}^{n_1} \frac{R_j}{s - \lambda_j}, \quad R_j = (\mathbf{C}\mathbf{x}_j)(\mathbf{y}_j^*\mathbf{B}),$$

while the constant contribution of the poles at ∞ is $-\mathbf{C}_2\mathbf{B}_2$. Finally, according to (3.5) this is also the first Markov parameter of the fast subsystem:

$$\mu_0 = -\mathbf{C}_2\mathbf{B}_2 = -(\mathbf{C}\mathbf{X}_\infty)(\mathbf{Y}_\infty^*\mathbf{A}\mathbf{X}_\infty)^{-1}(\mathbf{Y}_\infty^*\mathbf{B}). \quad (3.15)$$

Note that if $\mathbf{H}(s)$ is strictly proper, then $R_\infty = -\mathbf{C}_2\mathbf{B}_2 = \lim_{s \rightarrow \infty} \mathbf{H}(s) = 0$. ■

To the author's best knowledge, with modal approximation only the strictly proper part of the transfer function is reduced, interpolating it at the *finite* dominant poles. This neglects however possible non-zero contributions of poles at ∞ , and affects the response of the reduced system, as observed in Examples 3.2 and 3.4 below. Lemma 3.1 suggests that this contribution is recovered by constructing projection matrices which also include eigenvectors corresponding to infinite poles, along with the eigenvectors corresponding to dominant poles. From (3.15) however, it is clear that for large-scale problems, the subspaces spanned by the infinite eigenvectors $\mathbf{Y}_\infty$ and $\mathbf{X}_\infty$ may be impossible to compute. Nonetheless, these subspaces can be cheaply approximated when only few of the poles at ∞ are dominant, as is proposed next. Similarly to [34], consider the expansion of $\mathbf{B}$ and $\mathbf{C}$ in the basis of eigenvectors of

($\mathbf{A}, \mathbf{E}$) as follows:

$$\mathbf{B} = \sum_{j=1}^{n_1} \beta_j \mathbf{E} \mathbf{x}_j + \sum_{i=1}^{n_2} \beta_{\infty_i} \mathbf{A} \mathbf{x}_{\infty_i} \quad (3.16)$$

$$\mathbf{C} = \sum_{j=1}^{n_1} \gamma_j \mathbf{y}_j^* \mathbf{E} + \sum_{i=1}^{n_2} \gamma_{\infty_i} \mathbf{y}_{\infty_i}^* \mathbf{A} \quad (3.17)$$

The expansion coefficients in the basis of eigenvectors corresponding to the finite poles are found as usual from $\beta_j = \mathbf{y}_j^* \mathbf{B}$, $\gamma_j = \mathbf{C} \mathbf{x}_j$. These coefficients determine the degree of controllability and observability of the corresponding states in the new basis [5]. Accordingly, the pole λ_j for which $\frac{\gamma_j \beta_j}{\text{Re}(\lambda_j)}$ is large is called dominant [15, Chapter 3]. This is the basis of modal approximation, where components of the transfer function corresponding to most dominant poles are kept in the reduced transfer function. Similarly, the expansion coefficients in the basis of eigenvectors corresponding to infinite eigenvalues are found using (3.18). A dominance criterion for poles λ_i at ∞ is thus established.

Definition 3.1 A pole of $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ at $\lambda_i = \infty$ is dominant if $\beta_{\infty_i} \neq 0$ and $\gamma_{\infty_i} \neq 0$, where (3.18).

$$\beta_{\infty_i} = \frac{\mathbf{y}_{\infty_i}^* \mathbf{B}}{\mathbf{y}_{\infty_i}^* \mathbf{A} \mathbf{x}_{\infty_i}}, \quad \gamma_{\infty_i} = \frac{\mathbf{C} \mathbf{x}_{\infty_i}}{\mathbf{y}_{\infty_i}^* \mathbf{A} \mathbf{x}_{\infty_i}} \quad i \leq n_2. \quad (3.18)$$

It turns out that in most circuit simulation applications, only one or two of the poles at ∞ are dominant. This is linked to the structure of the input and output matrices $\mathbf{B}$ and $\mathbf{C}$. For many circuit simulation examples $\mathbf{B}$ and $\mathbf{C}$ are vectors with one or two non-zero entries and additionally often satisfy $\mathbf{B} \in \text{Null}(\mathbf{E})$, $\mathbf{C}^* \in \text{Null}(\mathbf{E}^*)$ (this holds for instance for the transmission line models in Figures 2.6 and 2.4). The infinite subspaces $\mathbf{Y}_{\infty}$ and $\mathbf{X}_{\infty}$ are thus easily approximated, as Theorem

3.1 demonstrates. First, a few technical remarks are in place. Note that $\mathbf{y}_{\infty_i}^* \mathbf{A} \mathbf{x}_{\infty_i} = \lambda_i \mathbf{y}_{\infty_i}^* \mathbf{E} \mathbf{x}_{\infty_i}$ where $\lambda_i = \infty \Rightarrow \frac{1}{\lambda_i} \mathbf{y}_{\infty_i}^* \mathbf{A} \mathbf{x}_{\infty_i} = 0$. Therefore, $\frac{1}{\lambda_i} = 0$ is an eigenvalue of $\mathbf{E}$ with $\mathbf{x}_{\infty_i}$, $\mathbf{y}_{\infty_i}$ being the corresponding right/left eigenvectors. This implies that $\mathbf{X}_{\infty}$ spans $\text{Null}(\mathbf{E})$ and $\mathbf{Y}_{\infty}$ spans $\text{Null}(\mathbf{E}^*)$. The singularity of $\mathbf{E}$ in most circuit simulation applications is due to the structure of $\mathbf{E}$, which has a non-singular block banded by zero rows and columns. Thus one can assume without loss of generality that the right/left eigenvectors of $(\mathbf{A}, \mathbf{E})$, $\mathbf{x}_i$, $\mathbf{y}_i$, $i \leq n_2$, corresponding to $\lambda_i = \infty$ are orthonormal. Consequently $\mathbf{X}_{\infty}^* \mathbf{X}_{\infty} = \mathbf{I}_{n_2}$ and $\mathbf{Y}_{\infty}^* \mathbf{Y}_{\infty} = \mathbf{I}_{n_2}$. Furthermore, as $\mathbf{E}$ is square, we know $\text{Null}(\mathbf{E}) = \text{Null}(\mathbf{E}^*)$ and so $\mathbf{Y}_{\infty} = \mathbf{X}_{\infty}$. We are ready to prove the following:

Theorem 3.1

Let $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ be a descriptor system with $\mathbf{E}$ singular, where $(\mathbf{A}, \mathbf{E})$ has n_2 eigenvalues at ∞ . Assume $\mathbf{B} \in \text{Null}(\mathbf{E})$, $\mathbf{C}^* \in \text{Null}(\mathbf{E}^*)$. Then $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ has one dominant pole at ∞ . Furthermore, if $\mathbf{CAB} \neq 0$ the dominant pole is interpolated by approximating subspaces $\mathbf{X}_{\infty}$ and $\mathbf{Y}_{\infty}$ with $\mathbf{B}$ and $\mathbf{C}$ respectively in (3.15).

Proof 3.2 Recall that $\mathbf{Y}_{\infty} = [\mathbf{y}_{\infty_1}, \dots, \mathbf{y}_{\infty_{n_2}}]$ and $\mathbf{X}_{\infty} = [\mathbf{x}_{\infty_1}, \dots, \mathbf{x}_{\infty_{n_2}}]$ both span $\text{Null}(\mathbf{E})$, as $\mathbf{E}$ is square. Since $\mathbf{B} \in \text{Null}(\mathbf{E}) \Rightarrow \mathbf{Y}_{\infty}^* \mathbf{B} = (0, \dots, 0, *, 0, \dots, 0)^T$. Similarly, since $\mathbf{C}^* \in \text{Null}(\mathbf{E}^*) \Rightarrow \mathbf{C} \mathbf{X}_{\infty} = (0, \dots, 0, *, 0, \dots, 0)$. Thus there is only one non-zero coefficient pair $(\beta_{\infty}, \gamma_{\infty})$ in (3.18) and according to Definition 3.1 only one dominant pole at ∞ . Consequently, only the dominant infinite pole (and corresponding left/right eigenvectors $\mathbf{y}_{\infty}$, $\mathbf{x}_{\infty}$) contributes to the constant term (3.15). Projecting $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ with $\mathbf{y}_{\infty}$, $\mathbf{x}_{\infty}$ is sufficient to capture this contribution, instead of using the full subspaces $\mathbf{Y}_{\infty}$ and $\mathbf{X}_{\infty}$. As $\mathbf{C} \in \text{Null}(\mathbf{E}^*)$ and $\mathbf{B} \in \text{Null}(\mathbf{E})$, we have the freedom to set $\mathbf{y}_{\infty} = \mathbf{C}$ and $\mathbf{x}_{\infty} = \mathbf{B}$. Provided $\mathbf{CAB} \neq 0$, projecting $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ with $\mathbf{C}$ and $\mathbf{B}$ interpolates the dominant pole at ∞ and recovers the

constant contribution $-\mathbf{C}_2\mathbf{B}_2 = -\mathbf{CB}(\mathbf{CAB})^{-1}\mathbf{CB}$. From Lemma 3.1, this is also the first Markov parameter of the fast subsystem $\Sigma(0, \mathbf{I}_{n_2}, \mathbf{B}_2, \mathbf{C}_2, 0)$. ■

Remark: When $\mathbf{CAB} = 0$, projecting with $\mathbf{C}$ and $\mathbf{B}$ would give a fast subsystem of the form $\Sigma(0, 0, \mathbf{B}_2, \mathbf{C}_2, 0)$. In some experiments however, this occurs jointly with $\mathbf{AB} \in \text{Null}(\mathbf{E})$ and $(\mathbf{CA})^* \in \text{Null}(\mathbf{E}^*)$. Thus one can approximate: $\mathbf{Y}_\infty^*$ with $[\mathbf{C}, \mathbf{CA}]$, and $\mathbf{X}_\infty$ with $[\mathbf{B}, \mathbf{AB}]$ and similar derivations for the dominance of two poles at ∞ can be developed.

3.2.2 Dominant SZM and poles at ∞

We saw how modal approximation can be enhanced with interpolation of dominant poles at ∞ , so that possible non-zero contributions of such poles are incorporated in the reduced transfer function. Extending the approach to the spectral zero interpolation method is straightforward:

Corollary 3.1

Given $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ with $\mathbf{E}$ singular, where $\mathbf{Y}_\infty, \mathbf{X}_\infty$ span $\text{Null}(\mathbf{E}^)$ and $\text{Null}(\mathbf{E})$ respectively. Let $\mathbf{W}$ and $\mathbf{V}$ be the projecting matrices from (2.12) which interpolate the dominant spectral zeros. Projecting $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ with $\mathbf{Y} = [\mathbf{W}, \mathbf{Y}_\infty]$ and $\mathbf{X} = [\mathbf{V}, \mathbf{X}_\infty]$ matches in addition the Markov parameter of the fast subsystem $\mu_0 = -\mathbf{C}_2\mathbf{B}_2$ in (3.5). Furthermore, when $\mathbf{B} \in \text{Null}(\mathbf{E})$, $\mathbf{C}^* \in \text{Null}(\mathbf{E}^*)$, $\mathbf{CAB} \neq 0$, $\mathbf{Y}_\infty, \mathbf{X}_\infty$ are replaced by $\mathbf{C}$ and $\mathbf{B}$ respectively and $\mu_0 = -\mathbf{CB}(\mathbf{CAB})^{-1}\mathbf{CB}$.*

Proof 3.3 From $\mathbf{E}\mathbf{X}_\infty = 0$ and $\mathbf{Y}_\infty^* \mathbf{E} = 0$, the projected matrices are:

$$\begin{aligned}
 \mathbf{Y}^* \mathbf{E} \mathbf{X} &= \begin{pmatrix} \mathbf{W}^* \mathbf{E} \mathbf{V} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{E}} & 0 \\ 0 & 0 \end{pmatrix} \\
 \mathbf{Y}^* \mathbf{A} \mathbf{X} &= \begin{pmatrix} \mathbf{W}^* \mathbf{A} \mathbf{V} & \mathbf{W}^* \mathbf{A} \mathbf{X}_\infty \\ \mathbf{Y}_\infty^* \mathbf{A} \mathbf{V} & \mathbf{Y}_\infty^* \mathbf{A} \mathbf{X}_\infty \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{A}} & \hat{\mathbf{A}}_{12} \\ \hat{\mathbf{A}}_{21} & \hat{\mathbf{A}}_{22} \end{pmatrix} \\
 \mathbf{Y}^* \mathbf{B} &= \begin{pmatrix} \mathbf{W} \mathbf{B} \\ \mathbf{Y}_\infty^* \mathbf{B} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{B}} \\ \hat{\mathbf{B}}_2 \end{pmatrix} \\
 \mathbf{C} \mathbf{X} &= (\mathbf{C} \mathbf{V}, \mathbf{C} \mathbf{X}_\infty) = (\hat{\mathbf{C}}, \hat{\mathbf{C}}_2)
 \end{aligned} \tag{3.19}$$

The projected system can be rewritten in compact form:

$$\begin{aligned}
 \hat{\mathbf{E}} \dot{\mathbf{x}}_1(t) &= [\hat{\mathbf{A}} - \hat{\mathbf{A}}_{12} \hat{\mathbf{A}}_{22}^{-1} \hat{\mathbf{A}}_{21}] \mathbf{x}_1(t) + [\hat{\mathbf{B}} - \hat{\mathbf{A}}_{12} \hat{\mathbf{A}}_{22}^{-1} \hat{\mathbf{B}}_2] \mathbf{u}(t) \\
 \mathbf{y}(t) &= [\hat{\mathbf{C}} - \hat{\mathbf{C}}_2 \hat{\mathbf{A}}_{22}^{-1} \hat{\mathbf{A}}_{21}] \mathbf{x}_1(t) - [\hat{\mathbf{C}}_2 \hat{\mathbf{A}}_{22}^{-1} \hat{\mathbf{B}}_2] \mathbf{u}(t).
 \end{aligned} \tag{3.20}$$

The constant term is $-\hat{\mathbf{C}}_2 \hat{\mathbf{A}}_{22}^{-1} \hat{\mathbf{B}}_2 = \mu_0$ in (3.5). When $\mathbf{B} \in \text{Null}(\mathbf{E})$, $\mathbf{C}^* \in \text{Null}(\mathbf{E}^*)$, $\mathbf{CAB} \neq 0$, the proof for $\mu_0 = -\mathbf{CB}(\mathbf{CAB})^{-1}\mathbf{CB}$ follows as in Theorem 3.1. Recall also that as $\mathbf{D} = 0$, the reduced system $\hat{\Sigma}(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \mathbf{D})$ was lossless, but with (3.20) the poles of the reduced system are moved away from the $j\omega$ axis. ■

In summary, the usual modal approximation or dominant spectral zero projecting matrices can be extended with eigenvectors from the subspaces $\mathbf{X}_\infty$ and $\mathbf{Y}_\infty$ that correspond to dominant poles at ∞ , according to Definition 3.1. In particular, the extended projectors will be $\mathbf{W}_{k+p} = [\mathbf{W}, \mathbf{Y}_{\infty_p}]$, $\mathbf{V}_{k+p} = [\mathbf{V}, \mathbf{X}_{\infty_p}]$, where $\mathbf{W}$ and $\mathbf{V}$ are the usual modal approximation or dominant SZM projectors of column rank k . $\mathbf{Y}_{\infty_p}$, $\mathbf{X}_{\infty_p}$ of column rank $p \ll n_2$ span the dominant eigenspace of $(\mathbf{A}, \mathbf{E})$ associated

with dominant poles at ∞ . In some cases, these are approximated easily with $\mathbf{C}$ and $\mathbf{B}$ respectively. An alternative to be investigated in the future is to compute $\mathbf{Y}_{\infty}$, $\mathbf{X}_{\infty}$ approximately as eigenvectors of $\mathbf{E}$ associated with dominant eigenvalues at 0. The theory developed in this section is applied in Examples 3.2 and 3.3.

3.3 Equivalent form 2: decoupled equations

We show how moment matching at ∞ is achieved when $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ has sparsity structures arising often in circuit simulation. It is known [37] that for any descriptor system with $\text{rank}(\mathbf{E}) = q$, there exist nonsingular matrices $\mathbf{Q}$ and $\mathbf{P}$ such that $\mathbf{QEP} = \text{diag}(\mathbf{I}_q, 0)$. Denoting $\mathbf{E}_{11} = \mathbf{I}_q$, with the coordinate transformation $[\mathbf{x}_1^T, \mathbf{x}_2^T]^T = \mathbf{P}^{-1}\mathbf{x}$, $\mathbf{x}_1 \in \mathbb{R}^q$, $\mathbf{x}_2 \in \mathbb{R}^{n-q}$, the transformed matrices are:

$$\mathbf{QEP} = \begin{pmatrix} \mathbf{E}_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{QAP} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}, \quad \mathbf{QB} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix}, \quad (\mathbf{CP})^T = \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix} \quad (3.21)$$

Equations (1.1) are thus decoupled into:

$$\begin{aligned} \mathbf{E}_{11}\dot{\mathbf{x}}_1(t) &= \mathbf{A}_{11}\mathbf{x}_1(t) + \mathbf{A}_{12}\mathbf{x}_2(t) + \mathbf{B}_1\mathbf{u}(t) \\ 0 &= \mathbf{A}_{21}\mathbf{x}_1(t) + \mathbf{A}_{22}\mathbf{x}_2(t) + \mathbf{B}_2\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}_1\mathbf{x}_1(t) + \mathbf{C}_2\mathbf{x}_2(t). \end{aligned} \quad (3.22)$$

System matrices from circuit simulation are often in the form (3.21), where $\mathbf{E}_{11}$ need not be $\mathbf{I}_q$, rather it is non-singular. $\mathbf{A}_{22}$ is also assumed non-singular. When the underlying circuit has neither inductive nor capacitive coupling, $\mathbf{E}_{11}$ is diagonal, thus very sparse. With inductive or capacitive coupling, $\mathbf{E}_{11}$ becomes block diagonal, triangular and in some cases full.

In (3.22), the dynamics of the system is described by the first equation, while the second is algebraic. Solving for $\mathbf{x}_2(t) = [-\mathbf{A}_{22}^{-1}\mathbf{A}_{21}]\mathbf{x}_1(t) - [\mathbf{A}_{22}^{-1}\mathbf{B}_2]\mathbf{u}(t)$ from the second equation and replacing it in the first and third, reveals the compact form (3.23).

$$\begin{aligned}\bar{\mathbf{E}}\dot{\mathbf{x}}_1(t) &= \bar{\mathbf{A}}\mathbf{x}_1(t) + \bar{\mathbf{B}}\mathbf{u}(t) \\ \mathbf{y}(t) &= \bar{\mathbf{C}}\mathbf{x}_1(t) + \bar{\mathbf{D}}\mathbf{u}(t),\end{aligned}\tag{3.23}$$

where

$$\begin{aligned}\bar{\mathbf{E}} &= \mathbf{E}_{11}, \quad \bar{\mathbf{A}} = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}, \quad \bar{\mathbf{B}} = \mathbf{B}_1 - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{B}_2 \\ \bar{\mathbf{C}} &= \mathbf{C}_1 - \mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{A}_{21}, \quad \bar{\mathbf{D}} = -\mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{B}_2.\end{aligned}\tag{3.24}$$

Since $\bar{\mathbf{E}}$ is invertible, $(\bar{\mathbf{A}}, \bar{\mathbf{E}})$ has no infinite eigenvalues and the following holds:

Fact 3.1

Dominant SZM applied on $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ is equivalent to dominant SZM applied on the compact system $\bar{\Sigma}(\bar{\mathbf{E}}, \bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}})$. Furthermore, the Markov parameters (moments at $s_0 = \infty$) of $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ are computed from the compact matrices (3.24):

$$\eta_0 = \bar{\mathbf{D}}, \quad \eta_j = \bar{\mathbf{C}}(\bar{\mathbf{E}}^{-1}\bar{\mathbf{A}})^{j-1}\bar{\mathbf{E}}^{-1}\bar{\mathbf{B}}, \quad j > 0.$$

System (3.23) is reduced as usual with dominant SZM, with $\mathbf{V}$ and $\mathbf{W}$ as in (2.12) being the projection matrices that interpolate (3.23) at the dominant spectral zeros. When $\bar{\mathbf{D}} = 0$, projecting with $\mathbf{V}$ and $\mathbf{W}$ alone will give a lossless reduced system (see discussion in Sect. 2.3.2). Similar to Sect. 3.2, the remedy is to match a Markov parameter in addition to the interpolating the dominant spectral zeros. The approach

is summarized in the following proposition.

Proposition 1 Let $\bar{\Sigma}(\bar{\mathbf{E}}, \bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}})$ be in compact form (3.23) and assume $\bar{\mathbf{D}} = 0$. Let $\mathbf{V}$ and $\mathbf{W}$ be the projection matrices that interpolate (3.23) at the dominant spectral zeros (from (2.7) or (2.12)). With extended projection matrices $\bar{\mathbf{V}} = [\mathbf{V}, \bar{\mathbf{E}}^{-1}\bar{\mathbf{B}}] \in \mathbb{C}^{n \times (k+1)}$, $\bar{\mathbf{W}} = [\mathbf{W}, \bar{\mathbf{C}}^*] \in \mathbb{C}^{n \times (k+1)}$, the reduced system matches the first Markov parameter $\eta_1 = \bar{\mathbf{C}} \bar{\mathbf{E}}^{-1} \bar{\mathbf{B}}$ in addition to interpolating the dominant spectral zeros.

Proof 3.4 Let the reduced matrices be $\hat{\bar{\mathbf{E}}} = \bar{\mathbf{W}}^* \bar{\mathbf{E}} \bar{\mathbf{V}}$, $\hat{\bar{\mathbf{A}}} = \bar{\mathbf{W}}^* \bar{\mathbf{A}} \bar{\mathbf{V}}$, $\hat{\bar{\mathbf{B}}} = \bar{\mathbf{W}}^* \bar{\mathbf{B}}$, $\hat{\bar{\mathbf{C}}} = \bar{\mathbf{C}} \bar{\mathbf{V}}$, $\hat{\bar{\mathbf{D}}} = 0$. The first Markov parameter of the reduced system is:

$$\begin{aligned}
 \hat{\eta}_1 &= \hat{\bar{\mathbf{C}}} \hat{\bar{\mathbf{E}}}^{-1} \hat{\bar{\mathbf{B}}} = \bar{\mathbf{C}} [\mathbf{V}, \bar{\mathbf{E}}^{-1} \bar{\mathbf{B}}] (\bar{\mathbf{W}}^* \bar{\mathbf{E}} \bar{\mathbf{V}})^{-1} \bar{\mathbf{W}}^* \bar{\mathbf{B}} \\
 &= \bar{\mathbf{C}} [\mathbf{V}, \bar{\mathbf{E}}^{-1} \bar{\mathbf{B}}] (\bar{\mathbf{W}}^* [\bar{\mathbf{E}} \mathbf{V}, \bar{\mathbf{B}}])^{-1} \bar{\mathbf{W}}^* \bar{\mathbf{B}} \\
 &= \bar{\mathbf{C}} [\mathbf{V}, \bar{\mathbf{E}}^{-1} \bar{\mathbf{B}}] ([\star, \bar{\mathbf{W}}^* \bar{\mathbf{B}}])^{-1} \bar{\mathbf{W}}^* \bar{\mathbf{B}} \\
 &= \bar{\mathbf{C}} [\mathbf{V}, \bar{\mathbf{E}}^{-1} \bar{\mathbf{B}}] \mathbf{e}_{k+1} \\
 &= \bar{\mathbf{C}} \bar{\mathbf{E}}^{-1} \bar{\mathbf{B}} = \eta_1
 \end{aligned}$$

For the interpolation of dominant spectral zeros, the proof is analogous (see for instance [3]). ■

Due to $\bar{\mathbf{E}}$ invertible, the matching of Markov parameters is a known result, see [3] for instance. Compared to the approach in Sect. 3.2, no assumptions about $\mathbf{B}$ and $\mathbf{C}$ are necessary. Rather, when the system matrices are easily transformed to the equivalent form (3.21), moment matching at ∞ for a system with singular $\mathbf{E}$, was simplified to the usual moment matching of a compact subsystem (3.23) with non-singular $\bar{\mathbf{E}}$. This is done easily with no additional numerical work, by simply appending the projecting matrices with $\bar{\mathbf{E}}^{-1} \bar{\mathbf{B}}$, $\bar{\mathbf{C}}$ (and additionally with $\bar{\mathbf{E}}^{-1} \bar{\mathbf{A}} \bar{\mathbf{E}}^{-1} \bar{\mathbf{B}}$,

$\overline{\mathbf{C}} \mathbf{E}^{-1} \overline{\mathbf{A}}$ if two moments at ∞ should be matched). Examples 3.4 and 3.5 demonstrate the approach in practice.

3.3.1 Numerical examples

To support the theoretical developments in this chapter, several examples are given.

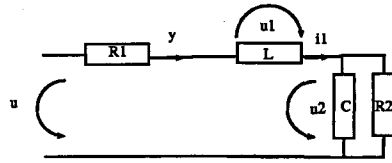


Figure 3.1 : RLC circuit with $\lim_{s \rightarrow \infty} \mathbf{H}(s) = 0$. All circuit elements have unit values.

Example 3.1

Consider the circuit in Fig. 3.1, with states $\mathbf{x} = [i_1, u_1, u_2]^T$, output $y = i_1$ and all circuit elements with unit values. The state matrices are:

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{C}^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (3.25)$$

The poles of the system are $\lambda_{1,2} = -1 \pm i$, $\lambda_3 = \infty$. The spectral zeros are $s_{1,2} = \pm 1.4142$, $s_{3...7} = \infty$. Due to the inductor in series with R_1 in the first loop, $\lim_{s \rightarrow \infty} \mathbf{H}(s) = 0$ and the transfer function $\mathbf{H}(s)$ is strictly proper. Thus, recalling the decomposition (3.5), the first Markov parameter of the fast subsystem should be 0, as is confirmed next. The equations are decoupled into slow and fast subsystems

of the form (3.1),(3.2) via the state transformations:

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.26)$$

The transformed matrices are:

$$\mathbf{QEP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{QAP} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{QB} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (\mathbf{CP})^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (3.27)$$

Thus according to the form (3.3), $\mathbf{B}_1^T = (1 \ 0)$, $\mathbf{B}_2 = -1$, $\mathbf{C}_1 = (1 \ 0)$, $\mathbf{C}_2 = 0$, $\mathbf{A}_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $\mathbf{N} = 0$, and the fast subsystem is:

$$0\dot{\mathbf{x}}_2(t) = \mathbf{x}_2(t) + \mathbf{B}_2\mathbf{u}(t), \quad \mathbf{y}_2(t) = 0. \quad (3.28)$$

From (3.5) the first Markov parameter of the fast subsystem is indeed $\mathbf{B}_2\mathbf{C}_2 = 0$.

Example 3.2

Consider the circuit in Fig. 4.2, with states chosen as $x_1 = u_1$, $x_2 = u_2$, $x_3 = i_2$, the current through L and $x_4 = i_1$, the current through R_1 , and output $\mathbf{y} = i_1$. Assuming all circuit elements have unit values, the system matrices are:

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{C}^T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (3.29)$$

The transfer function is $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \frac{s^3 + s^2 + 2s + 1}{s^3 + 2s^2 + 3s + 2}$. $\mathbf{H}(s)$ is proper, with $\lim_{s \rightarrow \infty} \mathbf{H}(s) = 1$. The system has poles at $\lambda_1 = -1$, $\lambda_{2,3} = -0.5 \pm 1.323i$, $\lambda_4 = \infty$, and the spectral zeros are: $s_{1,2} = 0.31454 \pm 1.3637i$, $s_{3,4} = -0.31454 \pm 1.3637i$, $s_{5,6} = \pm 0.72207$, $s_{7...9} = \infty$. Note that $\mathbf{EB} = 0$ and $\mathbf{CE} = 0$. The results with modal approximation and dominant SZM, with and without the moment matching of the fast subsystem at ∞ are shown in Fig. 3.2. For modal approximation, the chosen finite poles are $\lambda_{1,2}$, but the approximation misses the response at high frequencies. With additional moment matching at ∞ , the approximation is improved significantly. For dominant SZM spectral zeros $s_{5,6}$ are chosen. As discussed in Sect. 2.3.2, as $\mathbf{D} = 0$,

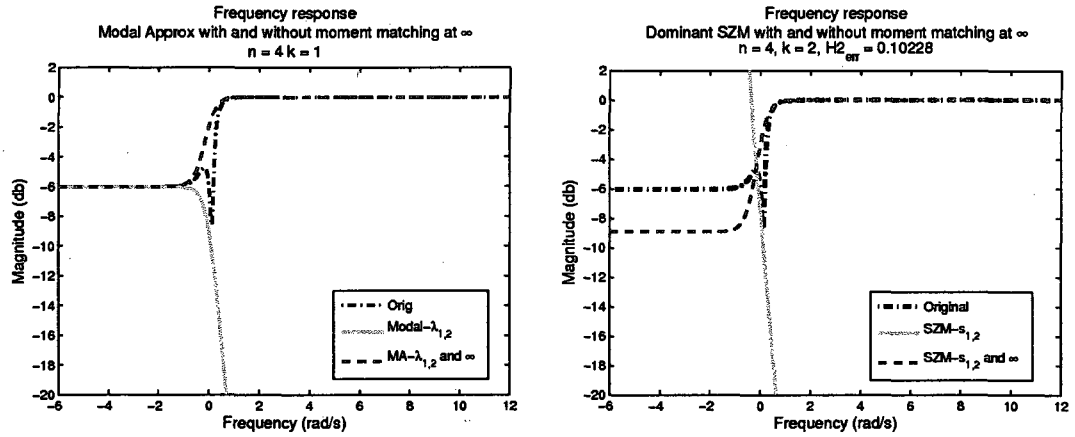


Figure 3.2 : Small RLC in Fig 4.2. Frequency response for modal approximation with and without moment matching at ∞ of the fast subsystem.

projecting with $\mathbf{V}$ and $\mathbf{W}$ that interpolate $s_{5,6}$, gives a lossless system, with a pole at 0 and all spectral zeros at ∞ . As with the enhanced modal approximation approach, this is resolved by projecting with $[\mathbf{W}, \mathbf{C}]$ and $[\mathbf{V}, \mathbf{B}]$ instead. The first Markov parameter of the fast subsystem is matched: $\mathbf{C}_2\mathbf{B}_2 = \mathbf{CB}(\mathbf{CAB})^{-1}\mathbf{CB} = 1$, the poles of the reduced system are -1.2 , ∞ and the spectral zeros are $s_{5,6}$ and three at ∞ .

Example 3.3

The transmission line models in Figures 2.4 and 2.6 both have associated systems $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ where $\mathbf{H}(s)$ is proper and $\lim_{s \rightarrow \infty} \mathbf{H}(s) \neq 0$. Furthermore they satisfy $\mathbf{BE} = 0$ and $\mathbf{CE} = 0$. Dominant SZM was applied with the additional moment matching at ∞ , as described in Sect. 3.2.2. For each transmission line, the approximation result was identical to what was already shown in Figures 2.5 and 2.8 respectively. Note that in Chapter 2, the lossless problem for $\mathbf{D} = 0$ was resolved via (2.21) in Sect. 2.3.2.

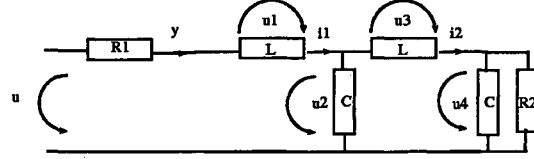


Figure 3.3 : RLC circuit with $\lim_{s \rightarrow \infty} \mathbf{H}(s) = 0$ for Example 3.4. All circuit elements have unit values.

Example 3.4

Consider the circuit in Fig. 3.3, with states chosen as $\mathbf{x} = [i_1, i_2, u_2, u_4, u_1, u_3]^T$ and output $\mathbf{y} = i_1$. The system matrices are:

$$\mathbf{E} = \begin{pmatrix} L & 0 & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{R_2} & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ R_1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{C}^T = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.30)$$

Already in the form (3.21), these matrices characterize a descriptor system of type (3.22). With all circuit elements having unit values, the transfer function is $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \frac{s^3 + s^2 + 2s + 1}{s^4 + 2s^3 + 4s^2 + 4s + 2}$. $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ has poles at $\lambda_{1,2} = -0.2571 \pm$

$1.5291i$, $\lambda_{3,4} = -0.7429 \pm 0.5291i$, $\lambda_{5,6} = \infty$ and spectral zeros $s_{1,2} = 0.3145 \pm 1.3637i$, $s_{3,4} = -0.3145 \pm 1.3637i$, $s_{5,6} = \pm 0.7221$, $s_{7...13} = \infty$. The system in compact form (3.23) has the same transfer function $\mathbf{H}(s) = \overline{\mathbf{H}}(s) = \overline{\mathbf{C}}(s\mathbf{E} - \overline{\mathbf{A}})^{-1}\overline{\mathbf{B}} + \overline{\mathbf{D}} = \frac{s^3+s^2+2s+1}{s^4+2s^3+4s^2+4s+2}$. Note that $\overline{\mathbf{D}} = 0$ and also $\lim_{s \rightarrow \infty} \mathbf{H}(s) = 0$, since $\mathbf{H}(s)$ is strictly proper.

Let $\overline{\Sigma}$ denote the system in compact form (3.23) and Σ the original system in the form (1.1). We reduce $\overline{\Sigma}$ with dominant SZM by constructing projection matrices $\mathbf{V}$ and $\mathbf{W}$ that interpolate at spectral zeros $s_{5,6}$. As $\overline{\mathbf{D}} = 0$, the reduced system is lossless, with transfer function $\frac{-0.7221}{-0.4178s}$, one pole at 0 and three spectral zeros at ∞ . Projecting $\overline{\Sigma}$ instead with $\overline{\mathbf{V}} = [\mathbf{V}, \overline{\mathbf{B}}]$, $\overline{\mathbf{W}} = [\mathbf{W}, \overline{\mathbf{C}}^*]$, the reduced system becomes strictly passive, with transfer function $\frac{1.416s+0.612}{1.416s^2+2.02s+1.7}$, stable poles $-0.7160 \pm 0.8331i$ and spectral zeros $s_{5,6} = \pm 0.7221$, $s_{7...9} = \infty$. A quick check verifies the matching of the first Markov parameter, according to Proposition 1: $\hat{\eta}_1 = \eta_1 = 1$. The frequency responses for the lossless and the strictly passive reduced systems is shown in Fig. 3.4, as compared to the frequency response of Σ . The modal approximant of Σ with interpolated dominant poles $\lambda_{3,4}$ is also shown.

The contrasting results for modal approximation without the moment matching at ∞ , for the Examples 3.2 and 3.4 are worth explaining further. In example 3.2, note that $\lim_{s \rightarrow \infty} \mathbf{H}(s) \neq 0$, and $\mathbf{H}(s)$ is proper. Any proper transfer function can be split into a strictly proper part and a constant term. Essentially, this is a special case of decomposition (3.4), where the polynomial part is only the constant term in (3.5), $\mathbf{B}_2\mathbf{C}_2 = \lim_{s \rightarrow \infty} \mathbf{H}(s) \neq 0$. Note however that the poles of the system are completely characterized by the strictly proper part of the transfer function and are independent of the constant term. As modal approximation only interpolates the poles of the system, it loses information about a non-zero constant term, and thus the response

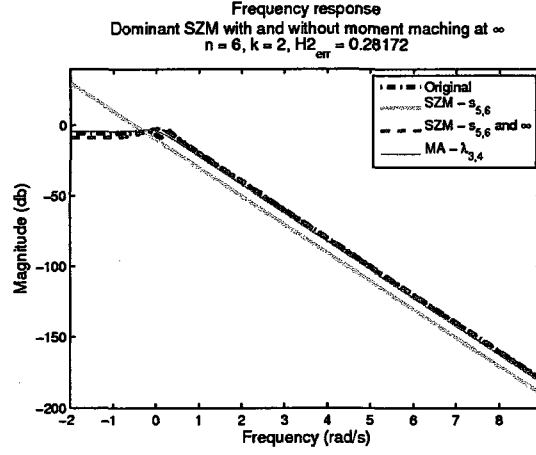


Figure 3.4 : RLC circuit in Fig. 3.3. Frequency response of original system and reduced with dominant SZM, dominant SZM with one moment matching at ∞ and modal approximation.

at ∞ is poorly approximated. Better matching for high frequencies with modal approximation is thus achieved by matching $\mathbf{B}_2\mathbf{C}_2 \neq 0$, the first Markov parameter of the fast subsystem, as discussed in Sect. 3.2 and shown in Fig. 3.2. In example 3.4 however, the transfer function was strictly proper and thus $\lim_{s \rightarrow \infty} \mathbf{H}(s) = 0$. As there is no constant term to account for, modal approximation captures the response by interpolating the dominant poles, and additional moment matching at ∞ is not necessary. This explains the quality of modal approximation in Fig. 3.4.

The properness of $\mathbf{H}(s)$ has a similar effect in the spectral zero interpolation method. Recalling the definition of spectral zeros (1.4), it is clear that a constant term in the transfer function will influence the value of the spectral zeros. The usual spectral zero interpolating projection becomes problematic when $\mathbf{D} = 0$, as the reduced poles are all on the $j\omega$ axis [28]. Similarly to modal approximation, the case $\lim_{s \rightarrow \infty} \mathbf{H}(s) \neq 0$ is resolved by matching the Markov parameter $\mathbf{B}_2\mathbf{C}_2 \neq 0$ in (3.5) as well (see Sect. 3.2). When $\lim_{s \rightarrow \infty} \mathbf{H}(s) = 0$ the approach was to match one Markov parameter, in addition to the interpolated dominant spectral zeros, of an equivalent

system in compact form (3.23) (see Sect. 3.3).

Example 3.5

Consider $N = 400$ LC sections in the circuit from Fig. 3.3 and states chosen as $\mathbf{x} = [u_1 \dots u_N, i_1 \dots i_N, u_{N+1} \dots u_{2N}]$ (voltages across capacitors, currents through inductors and voltages across inductors respectively), and output $\mathbf{y} = i_1$. The system is of the form $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$, with equations described by (3.22), easily convertible to (3.23). Furthermore, $\lim_{s \rightarrow \infty} \mathbf{H}(s) = 0$. In Fig. 3.5 we compare results for dominant SZM reduction with and without the moment matching at ∞ . Without the additional moment matching at ∞ , the reduced system is lossless. Modal approximation cannot capture the oscillations either, however dominant SZM with one moment matching at ∞ captures most of the response. It is worth comparing this result with the δ approach from Sect. 2.3.2, as shown in Fig. 3.6, left. The fact that the δ approach gives a poorer reduced model is expected, since it exploits the properties of modal approximation (see Sect. 2.3.2), which for this example was not accurate. Finally, a comparison with PRIMA is shown in Fig. 3.6, right.

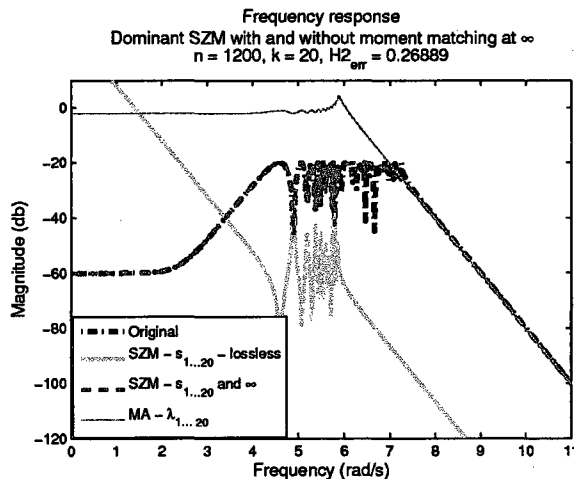


Figure 3.5 : Extended RLC transmission line in Fig. 3.3. Frequency response of original system and reduced with dominant SZM, dominant SZM with one moment matching at ∞ and modal approximation

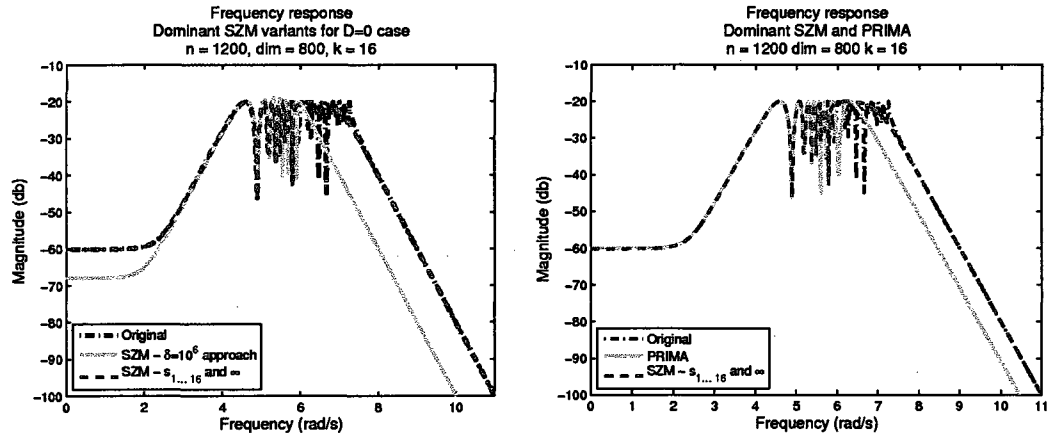


Figure 3.6 : Extended RLC transmission line in Fig. 3.3. Frequency response of original system and reduced with dominant SZM with one moment matching at ∞ , dominant SZM with the δ approach, and PRIMA.

3.4 Summary

This chapter showed how for systems $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ with certain matrix structures arising in circuit simulation, moment matching at ∞ can be used efficiently to improve reduction results with modal approximation and dominant SZM, even when $\mathbf{E}$ is singular. The combined method also resolves the lossless problem for the case $\mathbf{D} = 0$ in the context of dominant SZM. The approach is a possible solution for the future work mentioned in [36], which involves integrating rational Krylov methods into the partial realization framework for descriptor systems.

Chapter 4

Reduction of voltage transfer function

A method for reducing SISO transmission lines with respect to the voltage transfer function is proposed. The solution insures that the reduced voltage transfer function is underlying a reduced model realizable with RLC circuit components. Results with the proposed method are presented and compared with the MIMO approach for voltage transfer function reduction.

4.1 Background

Due to positive-realness requirements for the underlying transfer function, reduction for SISO systems describing transmission lines is generally performed with respect to the driving point impedance/admittance (where the output $\mathbf{y}$ is the voltage/current at the same port where input $\mathbf{u}$ is applied). Since transmission lines are more often used to transmit signals, rather than as driving point loads, reducing SISO transmission lines with respect to the voltage transfer is an emerging problem. This involves reduction of a non-positive real transfer function, where a different output $\mathbf{w}$ measures the voltage at the far end of the line, oppositely to $\mathbf{u}$.

According to [3, Theorem 5.26], passivity for electric circuits is equivalent to $\mathbf{H}(s)$ describing the network's driving point impedance (or admittance). Thus, by taking $\mathbf{u}$ and $\mathbf{y}$ at the same port, a network synthesis in terms of passive circuit elements is always possible. This is not guaranteed however, when an output $\mathbf{w}$ is measured

at a different location than $\mathbf{u}$, as the new transfer function is no longer positive real. Nevertheless, the circuit itself is still dissipative with respect to the supply rate (1.7). It remains to show how dissipativity is expressed in terms of the new output $\mathbf{w}$, as addressed this in Sect. 4.2.

Passive reduction with respect to a voltage transfer function can be done in a MIMO setting, by embedding the system in a two-input-two-output network, with $\mathbf{u}_1, \mathbf{y}_1$, the voltage and current at one port, and $\mathbf{u}_2, \mathbf{y}_2$ the current and voltage at the opposite port (see for instance Example 4.4.3). The resulting 2×2 transfer function is positive real. With MIMO dominant SZM reduction (see Sect. 4.4), a reduced positive real 2×2 transfer function is obtained, which can be further realized as a passive 2 port network. The (1, 2) entry of the MIMO reduced transfer function would correspond to the reduced voltage transfer function. Nevertheless, this thesis gives an alternative for directly reducing the SISO voltage transfer function of a transmission line, which only relies on the positive realness of the driving point transfer function and an additional output corresponding to the voltage transfer function. Even with the SISO approach, the circuit realization of the reduced model is still possible.

The approach is developed as follows. In Sect. 4.1.1 the difference between the two SISO reduction approaches is explained: driving point vs. opposite ends. Then, the theory underlying SISO voltage transfer function reduction is developed in Sect. 4.2, and numerical results follow in Sect. 4.3. The MIMO extension of dominant SZM is addressed in Sect. 4.4.

4.1.1 Driving point reduction

In the SISO case, the motivation for reducing the netlist with respect to the driving point admittance is the positive-realness of the underlying transfer function. In

particular, to guarantee that a reduced system is passive (thus realizable with RLC components), one needs to start from an original positive real transfer function. For RLC circuits, the original transfer function is guaranteed to be positive real by taking the output variable at the same port as where the input is applied. There is no guarantee of positive-realness however, when input and output variables are taken at opposite ports. This can be made clearer with through an example.

Example 4.1

Consider the simple circuit in Fig. 4.1. The state variables are $\mathbf{x}^T = [u_1, i_1]$, the voltage across capacitor C and current through inductor L respectively. The input voltage u is applied at the left port, and we measure the voltage across resistor R_2 , at the opposite port. The state matrices are:

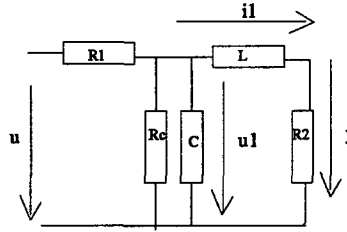


Figure 4.1 : Simple RLC circuit, with input voltage u at left port, and output variable y , voltage across resistor R_2 .

$$\mathbf{E} = \begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -\frac{1}{R_2} - \frac{1}{R_1} & -1 \\ 1 & -R_2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \frac{1}{R_1} \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{C}} = \begin{pmatrix} 0 & R_2 \end{pmatrix}, \quad \mathbf{D} = 0.$$

Assuming unit values for all RLC components, the transfer function is $\mathbf{H}(s) = \frac{1}{s^2 + 3s + 3}$, not positive real (the difference between the degrees of the numerator and denominator is 2, while for a positive real function it should be at most 1, [3, Theorem 5.22]). The system has spectral zeros on the $j\omega$ axis. A reduced order model obtained

directly from the voltage transfer function which underlies this system will be non-passive. If we measure instead the current at the input port, through resistor R_1 , the state matrices become:

$$\mathbf{C} = \begin{pmatrix} \frac{1}{R_1} & 0 \end{pmatrix}, \mathbf{D} = \frac{1}{R_1}. \quad (4.1)$$

The new transfer function is $\mathbf{H}(s) = \frac{s^2+2s+2}{s^2+3s+3}$ and is positive real. No spectral zeros are on the $j\omega$ axis and a passive-reduced order model can be obtained with SZM. Because the original input admittance transfer function is positive real, the reduced transfer function will also be positive real and the resulting system is passive, realizable with RLC circuit components.

4.2 Voltage transfer function reduction

In short, a passive, physically realizable reduced network is obtained first as usually, from the driving point reduction with the spectral zero interpolation method. The same projection is used afterwards to obtain the output which describes the reduced voltage transfer function. Thus, the original and reduced voltage transfer functions need not be positive-real themselves. Rather, their corresponding outputs are drawn from the original and reduced passive networks respectively, whose underlying driving point transfer functions are positive real. Next the theoretical foundation for the proposed method is developed.

Considering a SISO dynamical system of the form (1.1), where $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ describes the output measured at the input port, so that the resulting driving point transfer function (4.2) is positive real. $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is passive and realizable in terms of RLC circuit components. This also implies that Σ is dissipative with

respect to the supply rate (1.7). Consider also the output $\mathbf{w}(t) = \tilde{\mathbf{C}}\mathbf{x}(t)$, measuring any state (or linear combination of states) that is not at the input port, for instance the voltage at the far end of the transmission line. The associated non-positive real transfer function is (4.3). The two transfer functions are related by:

$$\mathbf{H}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \quad (4.2)$$

$$\tilde{\mathbf{H}}(s) = \frac{\mathbf{W}(s)}{\mathbf{U}(s)} = \tilde{\mathbf{C}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \quad (4.3)$$

$$\tilde{\mathbf{H}}(s) = \frac{\mathbf{W}(s)}{\mathbf{U}(s)} = \mathbf{H}(s) \underbrace{\frac{\mathbf{W}(s)}{\mathbf{Y}(s)}}_{\mathbf{R}(s)} = \mathbf{H}(s)\mathbf{R}(s). \quad (4.4)$$

The supply rate (1.7) can be expressed equivalently as a function of external variables $\mathbf{u}$ and $\mathbf{y}$, or as a function of external variables $\mathbf{u}$ and $\mathbf{w}$:

$$\mathbf{s}(\mathbf{u}, \mathbf{y}) = \underbrace{(\mathbf{y}^* \quad \mathbf{u}^*)}_{\mathbf{Q}} \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix}. \quad (4.5)$$

Using (4.4) and the fact that for exponential trajectories $\mathbf{u}(t) = e^{st}$, $\mathbf{y}(t) = \mathbf{H}(s)\mathbf{u}(t)$ and $\mathbf{w}(t) = \tilde{\mathbf{H}}(s)\mathbf{u}(t)$, we express the supply (1.7) in terms of the output $\mathbf{w}(t)$:

$$\begin{aligned} \mathbf{s} &= \mathbf{u}^*\mathbf{y} + \mathbf{y}^*\mathbf{u} = \mathbf{u}^*\mathbf{H}(s)\mathbf{u} + \mathbf{u}^*\mathbf{H}^*(s)\mathbf{u} = \mathbf{u}^*[\mathbf{H}(s) + \mathbf{H}^*(s)]\mathbf{u} \\ &= \mathbf{u}^* \left[\tilde{\mathbf{H}}(s) \frac{\mathbf{Y}(s)}{\mathbf{W}(s)} + \frac{\mathbf{Y}^*(s)}{\mathbf{W}^*(s)} \tilde{\mathbf{H}}^*(s) \right] \mathbf{u} = \mathbf{u}^*\mathbf{w} \frac{\mathbf{Y}(s)}{\mathbf{W}(s)} + \frac{\mathbf{Y}^*(s)}{\mathbf{W}^*(s)} \mathbf{w}^*\mathbf{u} \Rightarrow \end{aligned}$$

$$\mathbf{s}(\mathbf{u}, \mathbf{w}) = (\mathbf{w}^* \quad \mathbf{u}^*) \underbrace{\begin{pmatrix} 0 & \frac{\mathbf{Y}^*(s)}{\mathbf{W}^*(s)} \\ \frac{\mathbf{Y}(s)}{\mathbf{W}(s)} & 0 \end{pmatrix}}_{\mathbf{Q}_w} \begin{pmatrix} \mathbf{w} \\ \mathbf{u} \end{pmatrix}. \quad (4.6)$$

While $\mathbf{Q}$ in (4.5) is constant, a dynamical system underlies $\mathbf{Q}_w$ in (4.6). Taking $\mathbf{R}(s) = \frac{\mathbf{W}(s)}{\mathbf{Y}(s)} = \frac{\tilde{\mathbf{H}}(s)}{\mathbf{H}(s)}$, we recover the form of this dynamical system, denoted by Σ_R .*

Theorem 4.1

Given transfer functions $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ and $\tilde{\mathbf{H}}(s) = \tilde{\mathbf{C}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, the system Σ_R with transfer function $\mathbf{R}(s) = \frac{\tilde{\mathbf{H}}(s)}{\mathbf{H}(s)}$ has the minimal realization (4.7).

Thus $\mathbf{R}(s) = \tilde{\mathbf{C}}(s\mathbf{E} - \mathbf{A} + \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1}$, and

$$\Sigma_R = \left[\begin{array}{c|c} (\mathbf{E}, \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}) & \mathbf{B}\mathbf{D}^{-1} \\ \hline \tilde{\mathbf{C}} & 0 \end{array} \right]. \quad (4.7)$$

Proof 4.1 Let $\mathbf{G}(s) = \frac{1}{\mathbf{H}(s)}$, with system $\Sigma_G = \left[\begin{array}{c|c} (\mathbf{E}, \mathbf{A}_G) & \mathbf{B}_G \\ \hline \mathbf{C}_G & \mathbf{D}_G \end{array} \right]$, where:

$$\mathbf{A}_G = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}, \quad \mathbf{B}_G = \mathbf{B}\mathbf{D}^{-1}, \quad \mathbf{C}_G = -\mathbf{C}\mathbf{D}^{-1}, \quad \mathbf{D}_G = \mathbf{D}^{-1}. \quad \text{Thus :} \quad (4.8)$$

$$\begin{aligned} \mathbf{R}(s) &= \frac{\tilde{\mathbf{H}}(s)}{\mathbf{H}(s)} = \tilde{\mathbf{H}}(s)\mathbf{G}(s) = [\tilde{\mathbf{C}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}][\mathbf{C}_G(s\mathbf{E} - \mathbf{A}_G)^{-1}\mathbf{B}_G + \mathbf{D}_G] \\ &= \underbrace{\tilde{\mathbf{C}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{C}_G(s\mathbf{E} - \mathbf{A}_G)^{-1}\mathbf{B}_G}_{\Sigma_P} + \tilde{\mathbf{C}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{D}_G \end{aligned} \quad (4.9)$$

*For simplicity, we derive w.l.o.g. Σ_R associated with the transfer function $\mathbf{R}(s) = \frac{\mathbf{W}(s)}{\mathbf{Y}(s)}$ instead of $\frac{\mathbf{Y}(s)}{\mathbf{W}(s)}$. The inverse system corresponding to $\frac{1}{\mathbf{R}(s)}$ can then be derived using [15, Theorem 5.3.5].

The product system is $\Sigma_P = \left[\begin{array}{c|c} (\mathbf{E}_P, \mathbf{A}_P) & \mathbf{B}_P \\ \hline \mathbf{C}_P & 0 \end{array} \right]$, where:

$$\mathbf{E}_P = \begin{pmatrix} \mathbf{E} & 0 \\ 0 & \mathbf{E} \end{pmatrix}, \mathbf{A}_P = \begin{pmatrix} \mathbf{A} & \mathbf{B}\mathbf{C}_G \\ 0 & \mathbf{A}_G \end{pmatrix}, \mathbf{B}_P = \begin{pmatrix} 0 \\ \mathbf{B}_G \end{pmatrix}, \mathbf{C}_P = (\tilde{\mathbf{C}} \ 0). \quad (4.10)$$

To obtain a minimal realization for Σ_P , let $\mathbf{T} = \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ 0 & \mathbf{I} \end{pmatrix}$ be a similarity trans-

formation, with $\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ 0 & \mathbf{I} \end{pmatrix}$ and $\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$. Transform now matrices of Σ_P by

similarity: $\tilde{\mathbf{E}}_P = \mathbf{T}\mathbf{E}_P\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{E} & 0 \\ 0 & \mathbf{E} \end{pmatrix}$, $\tilde{\mathbf{A}}_P = \mathbf{T}\mathbf{A}_P\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A}_G \end{pmatrix}$, $\tilde{\mathbf{B}}_P =$

$\mathbf{T}\mathbf{B}_P = \begin{pmatrix} -\mathbf{B}_G \\ \mathbf{B}_G \end{pmatrix}$, $\tilde{\mathbf{C}}_P = \mathbf{C}_P\mathbf{T}^{-1} = (\tilde{\mathbf{C}} \ \tilde{\mathbf{C}})$. Plugging the transformed matrices

back into (4.9) and using (4.8), we obtain:

$$\begin{aligned} \mathbf{R}(s) &= \tilde{\mathbf{C}}_P(s\tilde{\mathbf{E}}_P - \tilde{\mathbf{A}}_P)^{-1}\tilde{\mathbf{B}}_P + \tilde{\mathbf{C}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{D}_G \\ &= (\tilde{\mathbf{C}} \ \tilde{\mathbf{C}}) \begin{pmatrix} (s\mathbf{E} - \mathbf{A})^{-1} & 0 \\ 0 & (s\mathbf{E} - \mathbf{A} + \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} \end{pmatrix} \begin{pmatrix} -\mathbf{B}\mathbf{D}^{-1} \\ \mathbf{B}\mathbf{D}^{-1} \end{pmatrix} + \\ &+ \tilde{\mathbf{C}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{D}^{-1} = \tilde{\mathbf{C}}(s\mathbf{E} - \mathbf{A} + \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1}. \end{aligned} \quad (4.11)$$

Theorem 4.1 together with (4.6) say that, even though $\tilde{\mathbf{H}}$ is not positive real, system $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \tilde{\mathbf{C}}, 0)$ is still dissipative with respect to the supply rate (1.7), and relates to the passive system $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ (with positive real $\mathbf{H}$) via (4.11). Thus we can use the passive $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ to obtain the passivity preserving projection via SZM, and use this projection to also reduce $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \tilde{\mathbf{C}}, 0)$. Clearly, the two

reduced systems will only differ in the corresponding reduced outputs, and as a result will also satisfy a relation of type (4.11), with the corresponding reduced matrices. The procedure is outlined next and supported with examples.

4.2.1 Procedure for SISO voltage transfer reduction

Given are the dynamical system $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with positive real transfer function (4.2) describing the driving point impedance, and $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \tilde{\mathbf{C}}, 0)$ with non-positive real transfer function (4.2) describing the voltage transfer function. To obtain a reduced system $\hat{\Sigma}(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\tilde{\mathbf{C}}}, 0)$ for the reduced voltage transfer function, that is also synthesizable in terms of RLC components, proceed as follows:

- 1) Build passivity preserving projections $\mathbf{V}$ and $\mathbf{W}$ according to (2.12), that interpolate $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ at the desired spectral zeros. The method has been described in Sect. 2.2.
- 2) Project $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \tilde{\mathbf{C}}, 0)$ with $\mathbf{V}$ and $\mathbf{W}$ from step 1).

$$\hat{\mathbf{E}} = \mathbf{W}^* \mathbf{E} \mathbf{V}, \quad \hat{\mathbf{A}} = \mathbf{W}^* \mathbf{A} \mathbf{V}, \quad \hat{\mathbf{B}} = \mathbf{W}^* \mathbf{B}, \quad \hat{\tilde{\mathbf{C}}} = \tilde{\mathbf{C}} \mathbf{V} \quad (4.12)$$

- 3) Project $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ according to (2.13). The reduced $\hat{\Sigma}(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \mathbf{D})$ is passive, with positive real driving point transfer function $\hat{\mathbf{H}}(s)$, thus it is synthesizable in terms of RLC components using [38], [39].
- 4) Note that the reduced states are independent of the output, i.e., $\dot{\hat{\mathbf{E}}}\hat{\mathbf{x}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t)$. Thus, from the circuit obtained in step 3), we can draw the output $\hat{\mathbf{w}} = \hat{\tilde{\mathbf{C}}}\hat{\mathbf{x}}(t)$ corresponding to the reduced voltage transfer function. The linear combination of reduced states underlying the reduced voltage transfer

function is therefore simply given by $\hat{\tilde{C}}$.

4.3 Numerical examples

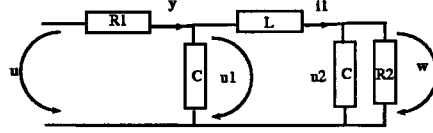


Figure 4.2 : Simple RLC circuit illustrating SISO voltage transfer function reduction, with input voltage u at left port, and output variables y and w . All circuit elements have unit values.

We illustrate the procedure through some examples.

Example 4.2

Consider the circuit in Fig. 4.2, with all circuit elements having unit values. With states chosen as the voltages across the two capacitors and the current through the inductor $\mathbf{x}(t) = [u_1, u_2, i_1]$, let y measure the current through R_1 , i.e., $y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$, $\mathbf{C} = [-\frac{1}{R_1}, 0, 0]$, $\mathbf{D} = \frac{1}{R_1}$. The transfer function corresponding to the driving point impedance is $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{s^3 + s^2 + 2s + 1}{s^3 + 2s^2 + 3s + 2}$, and is positive real. Let output $w = \tilde{\mathbf{C}}\mathbf{x}(t)$, $\tilde{\mathbf{C}} = [0, 1, 0]$ measure the voltage at the far end of the line, so the voltage transfer function is $\tilde{\mathbf{H}}(s) = \tilde{\mathbf{C}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \frac{s+1}{s^3 + 2s^2 + 3s + 2}$, non-positive-real. The spectral zeros of $\mathbf{H}(s)$ are: $s_{1,2} = 0.31454 \pm 1.3637i$, $s_{3,4} = -0.31454 \pm 1.3637i$, $s_{5,6} = \pm 0.72207$. In step 1), we construct the passivity preserving projectors that interpolate $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ at the spectral zeros $s_{1...4}$:

$$\mathbf{V} = [(s_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}, (s_1^*\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}] \quad (4.13)$$

$$\mathbf{W}^* = [(-s_1\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^*, (-s_1^*\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^*] \quad (4.14)$$

The projection (4.12) in step 2) gives $\hat{\mathbf{C}} = [-0.26421, 0.60613]$ and a reduced voltage transfer function $\hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s\hat{\mathbf{E}} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} = \frac{-0.4247s+1.147}{s^2+1.136s+2.315}$, non-positive real. The original and reduced voltage transfer functions $\tilde{\mathbf{H}}(s)$ and $\hat{\mathbf{H}}(s)$ are shown in Fig. 4.3. The reduced positive real transfer function corresponding to the input impedance is obtained from step 3) $\hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s\hat{\mathbf{E}} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} = \frac{s^2+0.3966s+1.657}{s^2+1.136s+2.315}$, where $\hat{\mathbf{C}} = [-0.4771, -0.19333]$. A reduced network can be synthesized from the positive real $\hat{\mathbf{H}}(s)$ using for instance [38]. Finally, in step 4) we draw the output of this reduced network according to $\hat{\mathbf{w}} = \hat{\mathbf{C}}\hat{\mathbf{x}}(t)$.

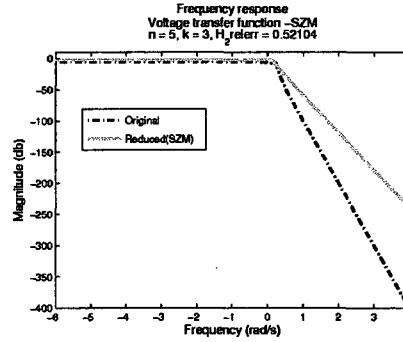


Figure 4.3 : Voltage transfer function reduction for circuit in Fig. 4.2.

Remark: Note that $\mathbf{R}(s) = \frac{\tilde{\mathbf{H}}(s)}{\hat{\mathbf{H}}(s)} = \frac{s+1}{s^3+s^2+2s+1}$ satisfies Theorem 4.1. Also $\hat{\mathbf{R}}(s) = \frac{\hat{\tilde{\mathbf{H}}}(s)}{\hat{\mathbf{H}}(s)} = \frac{-0.4247s+1.14}{s^2+0.3966s+1.657}$ and it is easily verified that $\hat{\mathbf{R}}(s) = \hat{\mathbf{C}}(s\hat{\mathbf{E}} - \hat{\mathbf{A}} + \hat{\mathbf{B}}\mathbf{D}^{-1}\hat{\mathbf{C}})^{-1}\hat{\mathbf{B}}\mathbf{D}^{-1}$.

Example 4.3

We apply the above SISO voltage transfer function reduction on the RLC transmission line in Fig. 2.6. The output $\mathbf{w}$ is measured as the voltage across the capacitor at the last port. The approximation is shown in Fig. 4.4.

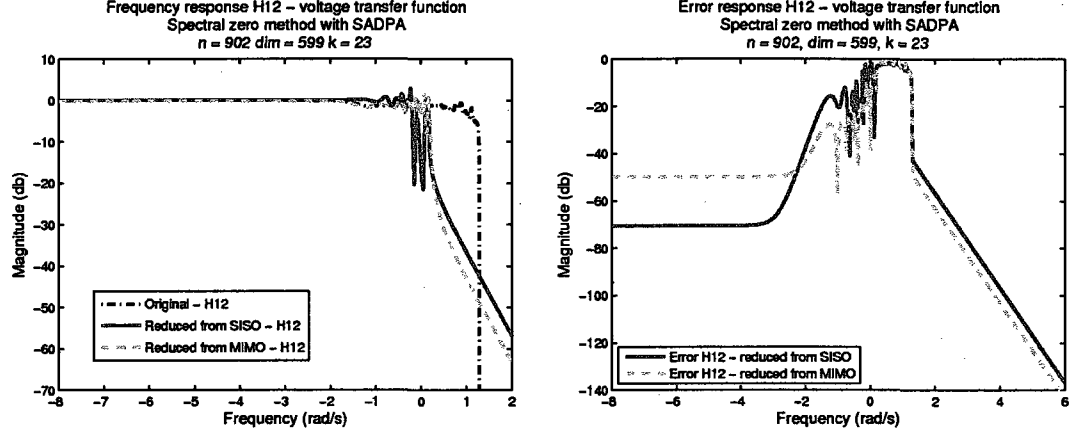


Figure 4.4 : RLC transmission line in Fig. 2.6. Voltage transfer function reduction with dominant SZM. Frequency response and error function.

4.4 MIMO Dominant SZM

This section presents reduction results with MIMO dominant SZM. The implementation is based on the *subspace accelerated MIMO dominant pole algorithm (SAMDP)* [15, Chapter 4], [19]. In what follows, Σ describes an original MIMO system with n states, m inputs and p outputs, and transfer function $\mathbf{H}(s)$ of size $(m \times p)$. $\hat{\Sigma}$ is the MIMO reduced system, with k states, m inputs and p outputs, and $\hat{\mathbf{H}}(s)$ of size $(m \times p)$. For overall approximation assessment, we compare singular value plots of $\mathbf{H}(s)$ and $\hat{\mathbf{H}}(s)$. Relative approximation errors are measured as $\frac{\|\mathbf{H}(j\omega) - \hat{\mathbf{H}}(j\omega)\|_2}{\|\mathbf{H}(j\omega)\|_2}$ for frequency sweeps ω . Original and reduced transfer functions from one input to one output are also compared individually (e.g., the response from input u_1 to output y_2 is the $(1, 2)$ entry of $\mathbf{H}(s)$, approximated by the $(1, 2)$ entry of $\hat{\mathbf{H}}(s)$).

4.4.1 Small example

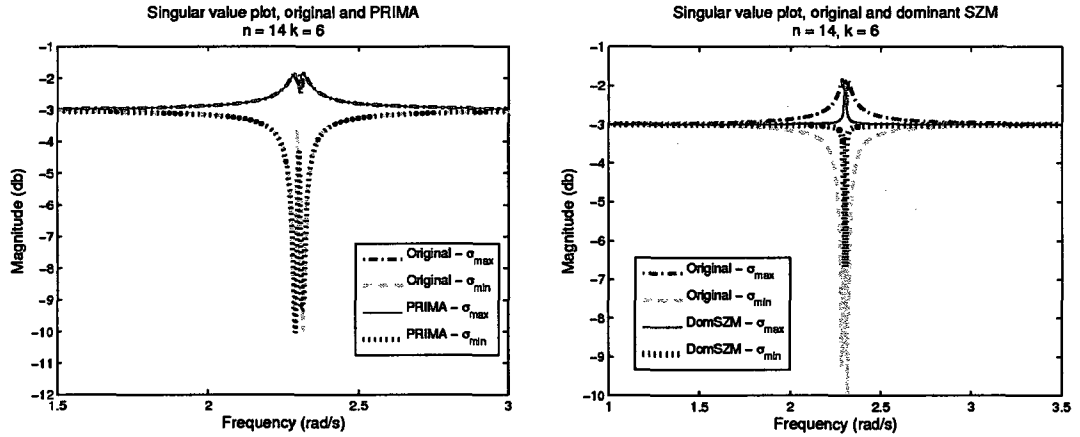


Figure 4.5 : MIMO RLC tline model. Singular value plots, original and reduced with dominant SZM and PRIMA.

We reduce a small MIMO system with $n = 14$ states, $m = 2$ inputs and $p = 2$ outputs, with MIMO dominant SZM and PRIMA. The singular value plots for the original and reduced systems are shown in Fig. 4.5. The transfer functions from each input to each output are shown in Fig. 4.6. The PRIMA model is more accurate.

4.4.2 MIMO transmission line, $n = 256$

A MIMO (2×2) transmission line model [40] was reduced. Fig. 4.7 shows the singular values plots of the original and reduced models. The system equations cannot be reformulated to satisfy the PRIMA requirements, thus the PRIMA reduced model is unstable, while with dominant SZM passivity is preserved. The interpolation effect at the dominant spectral zeros is reflected in the fact that dominant SZM captures more peaks in the σ_{max} plot than PRIMA. Dominant SZM reduces the individual transfer functions (input 1 to output 1), and (input 1 to output 2) as shown in Fig. 4.8. Table 4.1 summarizes the approximation performance with the two methods.

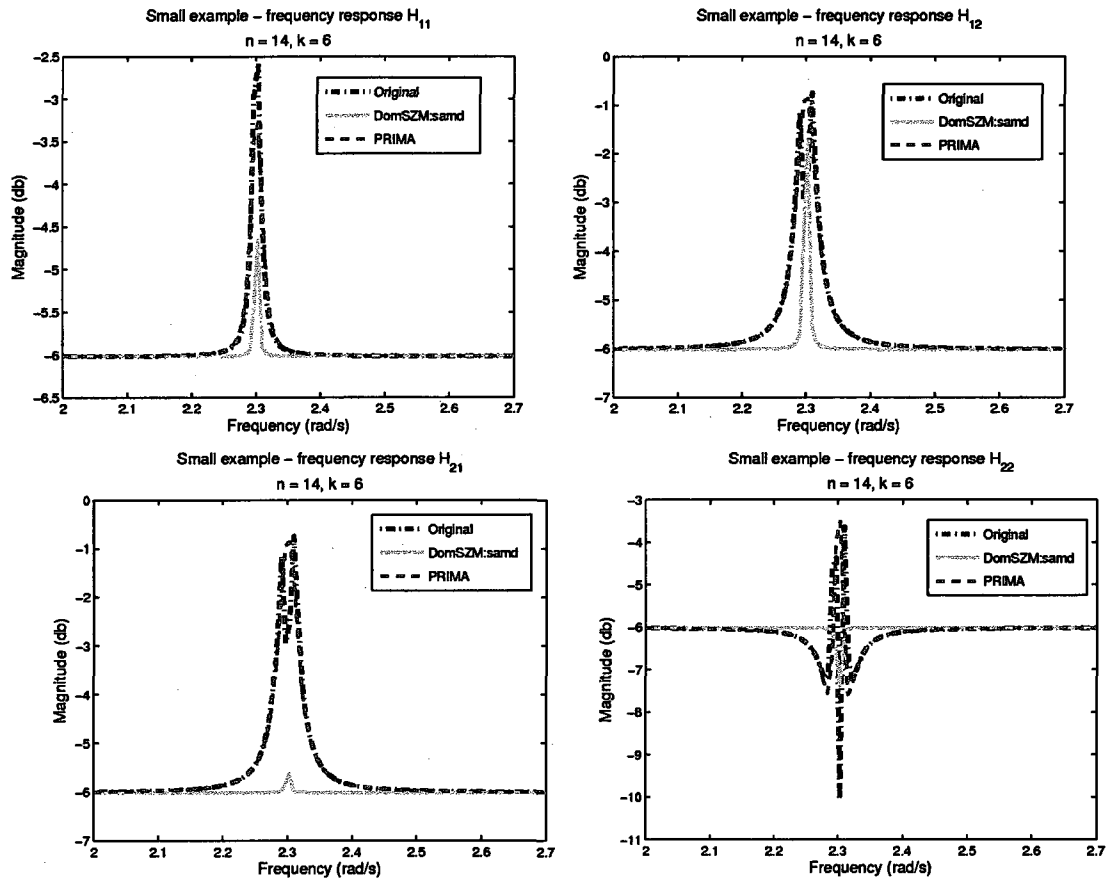


Figure 4.6 : MIMO small example. Individual transfer functions, original and reduced with dominant SZM and PRIMA.

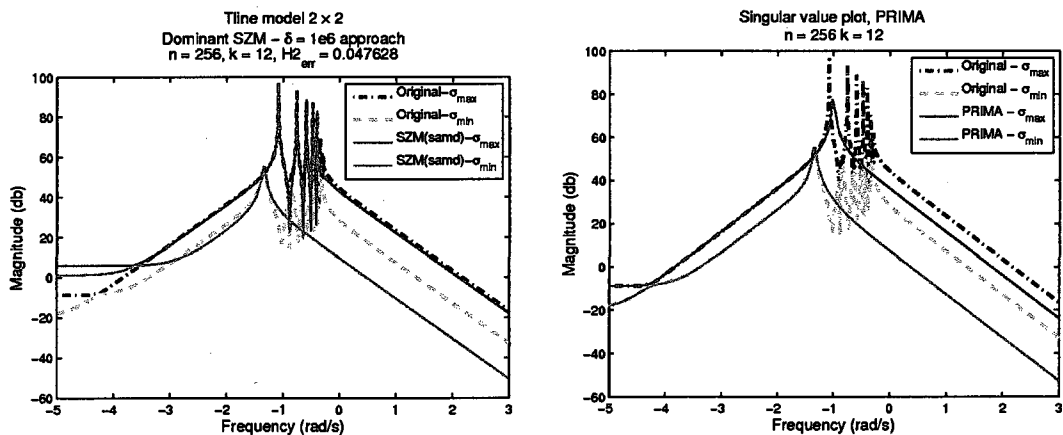


Figure 4.7 : MIMO 2×2 transmission line reduced with dominant SZM (left) and PRIMA (right-unstable). Singular value plots.

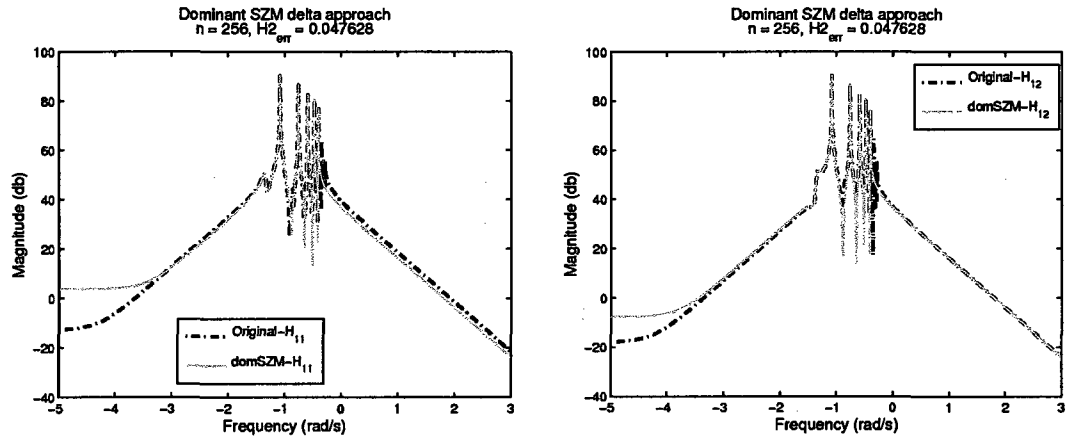


Figure 4.8 : MIMO 2×2 transmission line reduced with dominant SZM. Frequency response of individual SISO transfer functions: $\mathbf{H}(1,1)$ (left) and $\mathbf{H}(1,2)$ (right).

Table 4.1 : MIMO (2×2) tline, $n=256$ $k=12$, reduction summary

SADPA based	Error	Time(s)	Iterations
Dominant SZM	0.155	51	45 iter., 5 restarts
Other methods	Error	Time(s)	Constraints
PRIMA	0.975	0.1	unstable

4.4.3 MIMO version 1 of Fig. 2.6 model

We reduce the RLC transmission line in Fig. 2.6 by embedding it into a MIMO (2×2) transfer function as follows. A current source is added in parallel to R_C at the right port, defining the second input u_2 . The output y_2 is the voltage at the far end of the line, across the last R_C . The positive real MIMO (2×2) transfer function is reduced with dominant SZM and PRIMA. With PRIMA, the sign change is needed to convert $\mathbf{E}$ to definite, while dominant SZM applies directly. The singular value plots are compared in Fig. 4.9, and the individual transfer functions are shown in Fig. 4.10.

As with the SISO results in Sect. 2.4.3, the model reduced with dominant SZM

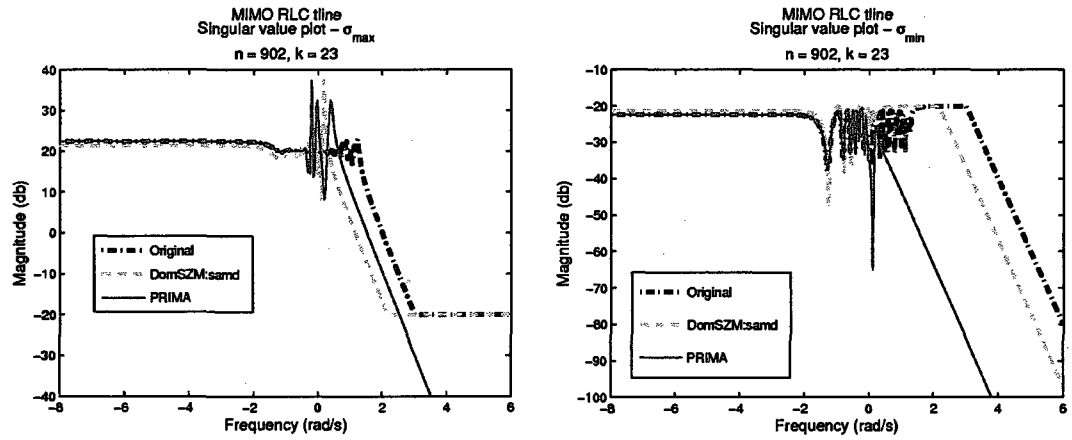


Figure 4.9 : RLC tline, MIMO model 1. Singular value plots, original and reduced with dominant SZM and PRIMA.

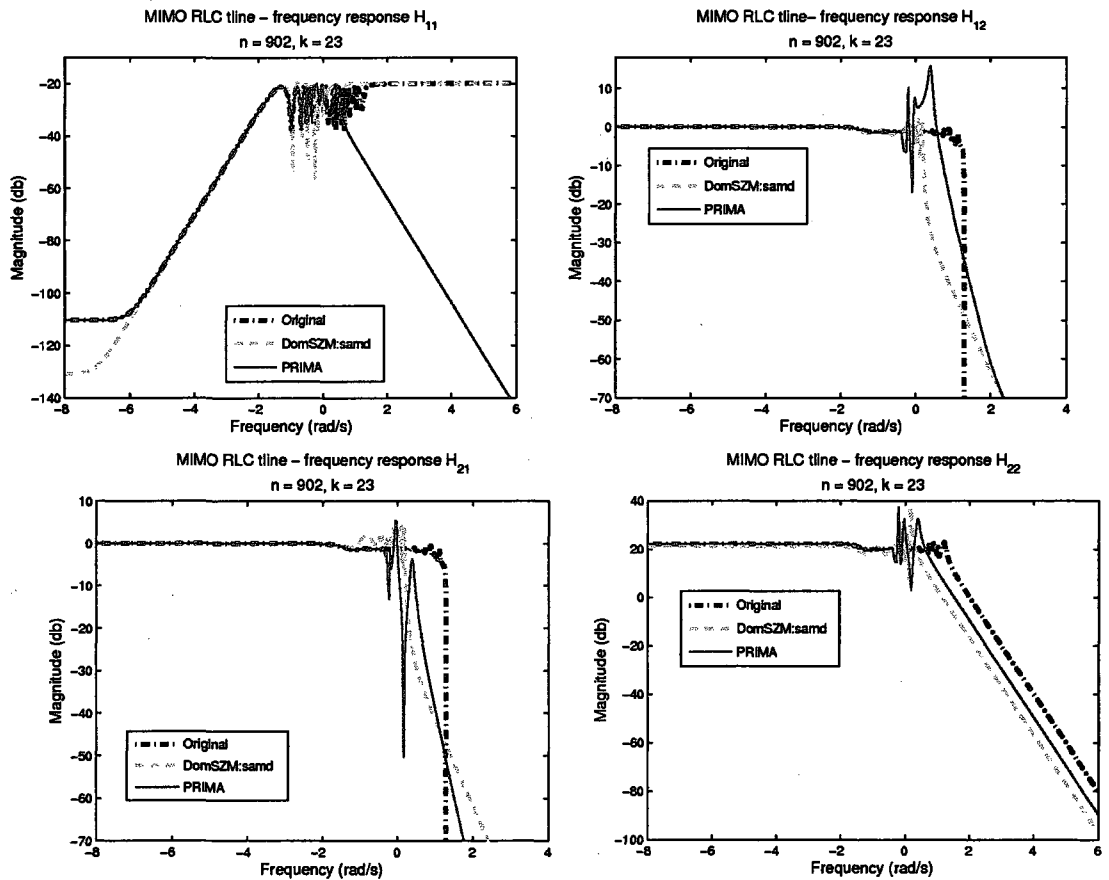


Figure 4.10 : RLC tline MIMO model 1, original and reduced with dominant SZM and PRIMA. Frequency response of SISO transfer functions.

Table 4.2 : RLC tline MIMO model 1, $n=902$ $k=23$, reduction summary

SADPA based	Error	Time(s)	Iterations
Dominant SZM	0.448	38.2	810 iter., 113 restarts
Other methods	Error	Time(s)	Constraints
PRIMA	0.471	24.9	E definite

is a better approximant than the PRIMA model for a wider frequency range. The reduction summary in Table 4.2 for the relative errors and computational time shows that dominant SZM is at least comparable to PRIMA. Although the computational time for obtaining a reduced model with dominant SZM is larger than with PRIMA, the quality of the reduced model is more important for simulation purposes. Dominant SZM also applies on the system matrices resulting from the simulator with no prior manipulations.

Finally, Fig. 4.4 compares the reduction of the voltage transfer function with the two approaches: the SISO procedure from Sect. 4.2.1, and the reduction starting from the MIMO system described here. Results with the two approaches are similar.

4.4.4 MIMO version 2 of Fig. 2.6 model

We reduce the RLC transmission line in Fig. 2.6, where an additional resistor R was added at the right end of the line, similarly to the left end of the line. Input voltage u_2 and output current y_2 (through the added resistor R) are defined. u_1 and y_1 at the left remain as in Fig. 2.6. The model becomes MIMO (2×2) and is reduced with dominant SZM and PRIMA. PRIMA needs the sign change to accommodate passivity, while dominant SZM applies directly on the system equations. Figures 4.11 and 4.12 show the singular value plots and the individual transfer functions respectively. PRIMA approximates the response better in the lower frequency range, while dominant SZM is a better approximant for higher frequencies. Table 4.3 summarizes the error and computational cost for both methods. Note that the MIMO PRIMA implementation becomes more expensive than MIMO dominant SZM, as the cost of orthogonalizing the Krylov vectors at each iteration inside PRIMA starts to dominate.

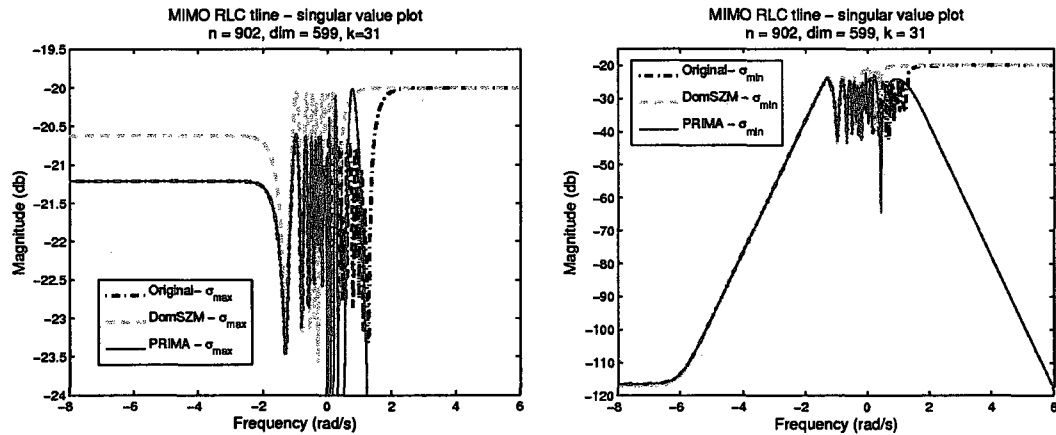


Figure 4.11 : MIMO RLC tline model. Singular value plots, original and reduced with dominant SZM and PRIMA.

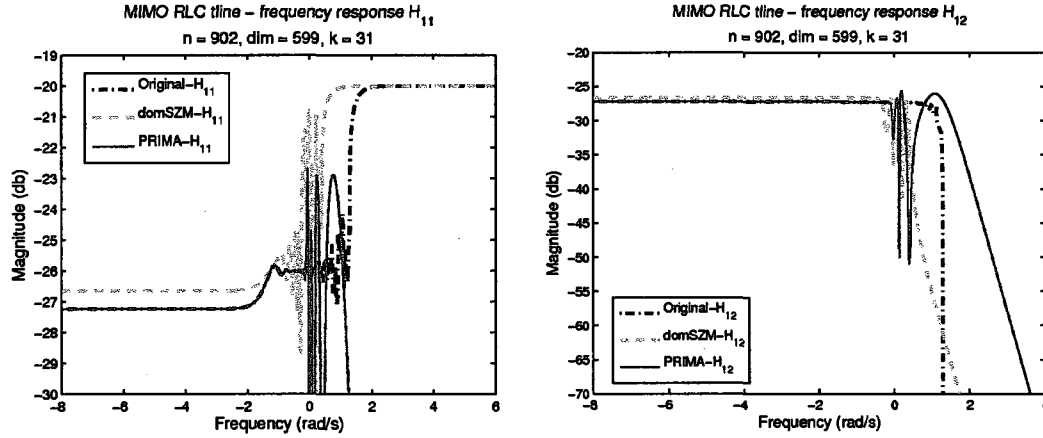


Figure 4.12 : MIMO RLC tline model, original and reduced with dominant SZM and PRIMA. Frequency response of SISO transfer functions: $\mathbf{H}(1,1)$ and $\mathbf{H}(1,2)$.

Table 4.3 : RLC tline MIMO model 2, $n=902$ $k=31$, reduction summary

SADPA based	Error	Time(s)	Iterations
Dominant SZM	0.171	13.54	197 iter., 25 restarts
Other methods	Error	Time(s)	Constraints
PRIMA	0.763	33.3	E definite

4.4.5 MIMO coupled RLC transmission line

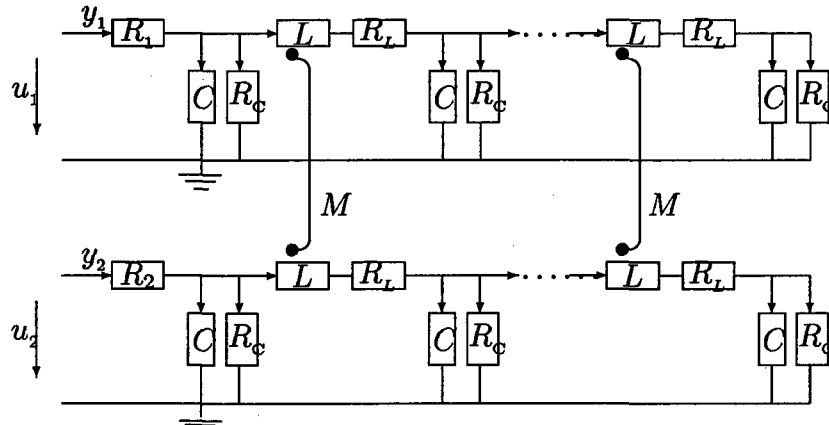


Figure 4.13 : Two coupled transmission lines

The inductively coupled MIMO RLC transmission line in Fig. 4.13 is reduced. The

system equations are written in the PRIMA-friendly "passive-form", and the system matrix $\mathbf{E}$ is invertible. From $n = 11998$, we reduced the system to dimension $k = 50$ with MIMO dominant SZM, PRIMA and modal approximation. Fig. 4.14 shows the plots for σ_{max} and σ_{min} of the original and reduced transfer functions. In Fig. 4.15, the individual components $\mathbf{H}(1,1)$ and $\mathbf{H}(1,2)$ of the original and reduced transfer functions are compared. Errors and CPU times are collected in Table 4.4. The PRIMA model is the most accurate and cheapest to compute. The Dominant SZM model is comparable to PRIMA, but misses some oscillations. Modal approximation misses most of the response. Even though results with PRIMA are most promising, they are strongly dependent on the original "passive form" for the system equations, and the fact that the pencil $(\mathbf{A}, \mathbf{E})$ has no eigenvalues at infinity.

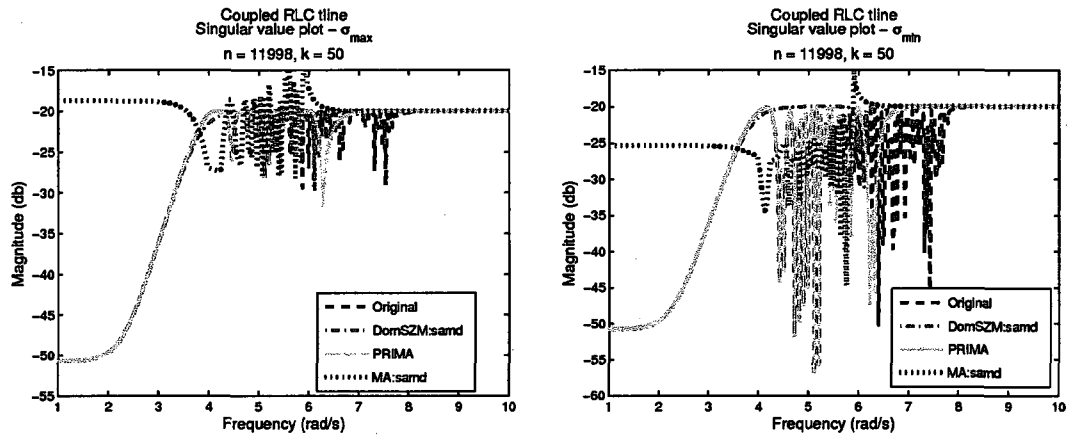


Figure 4.14 : MIMO coupled model in Fig. 4.13. Singular value plots, original and reduced with dominant SZM and PRIMA.

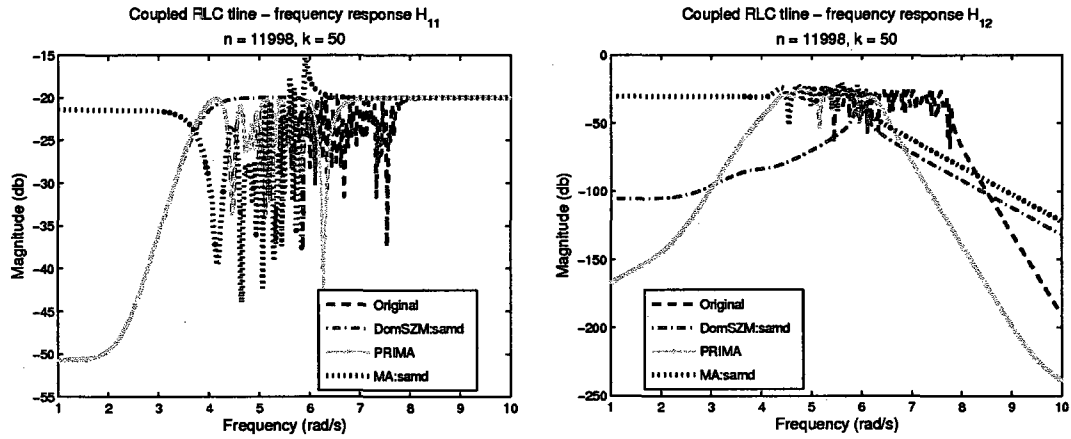


Figure 4.15 : MIMO coupled model in Fig. 4.13. Frequency response of SISO transfer functions: $H(1, 1)$ and $H(1, 2)$, original and reduced with dominant SZM and PRIMA.

Table 4.4 : MIMO coupled model in Fig. 4.13, $n=11998$ $k=50$, reduction summary

SADPA based	Error	Time(s)	Iterations
Dominant SZM	0.2823	226	163 iter., 18 restarts
Modal approx.	0.7136	125	239 iter., 26 restarts
Other methods	Error	Time(s)	<i>Constraints</i>
PRIMA	0.2047	8.4	-

4.4.6 MNA transmission line

We reduce an MNA transmission line model [40] with $n = 980$ states, $m = 4$ inputs and $n = 4$ outputs with modal approximation, dominant SZM and PRIMA. The dominant poles interpolated with modal approximation are shown in Fig. 4.16, left. With dominant SZM, the δ approach from Sect. 2.3.2 was used to resolve the lossless problem. The resulting poles and spectral zeros are as in Fig. 4.16, right. Fig. 4.17 shows the singular value plots. In Fig. 4.18, several individual transfer functions are compared. Overall, reduced models with dominant SZM and PRIMA approximate the response for the low to mid frequency range better than modal approximation. No method can approximate the response for higher frequencies.

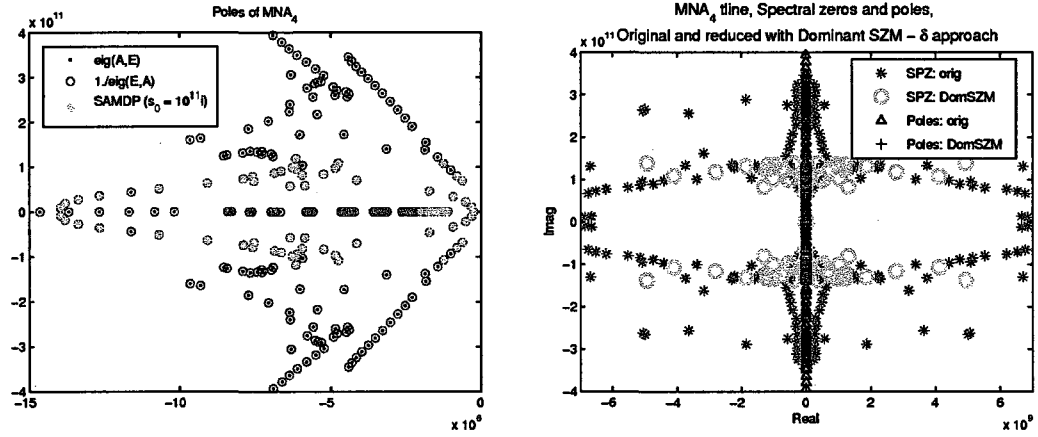


Figure 4.16 : MIMO MNA tline model. Dominant poles and spectral zeros.

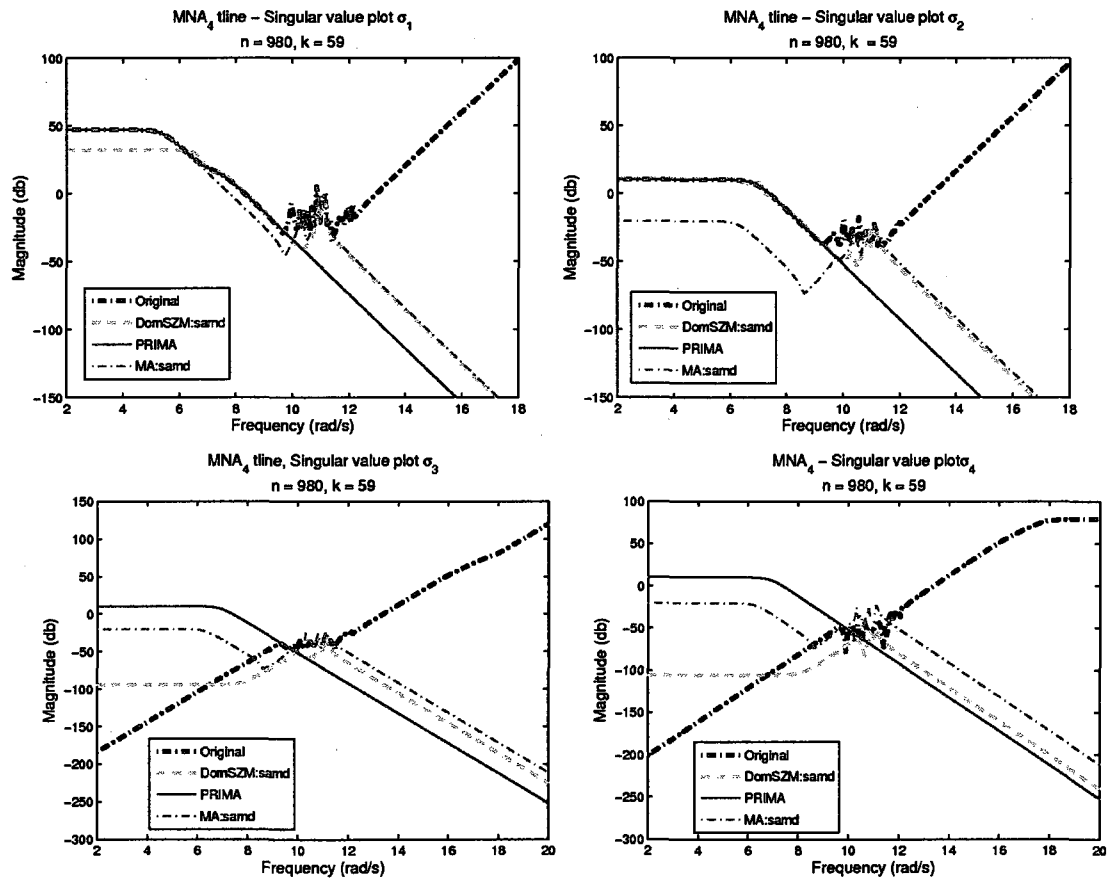


Figure 4.17 : MIMO MNA tline model. Singular value plots, original and reduced with dominant SZM, modal approximation and PRIMA.

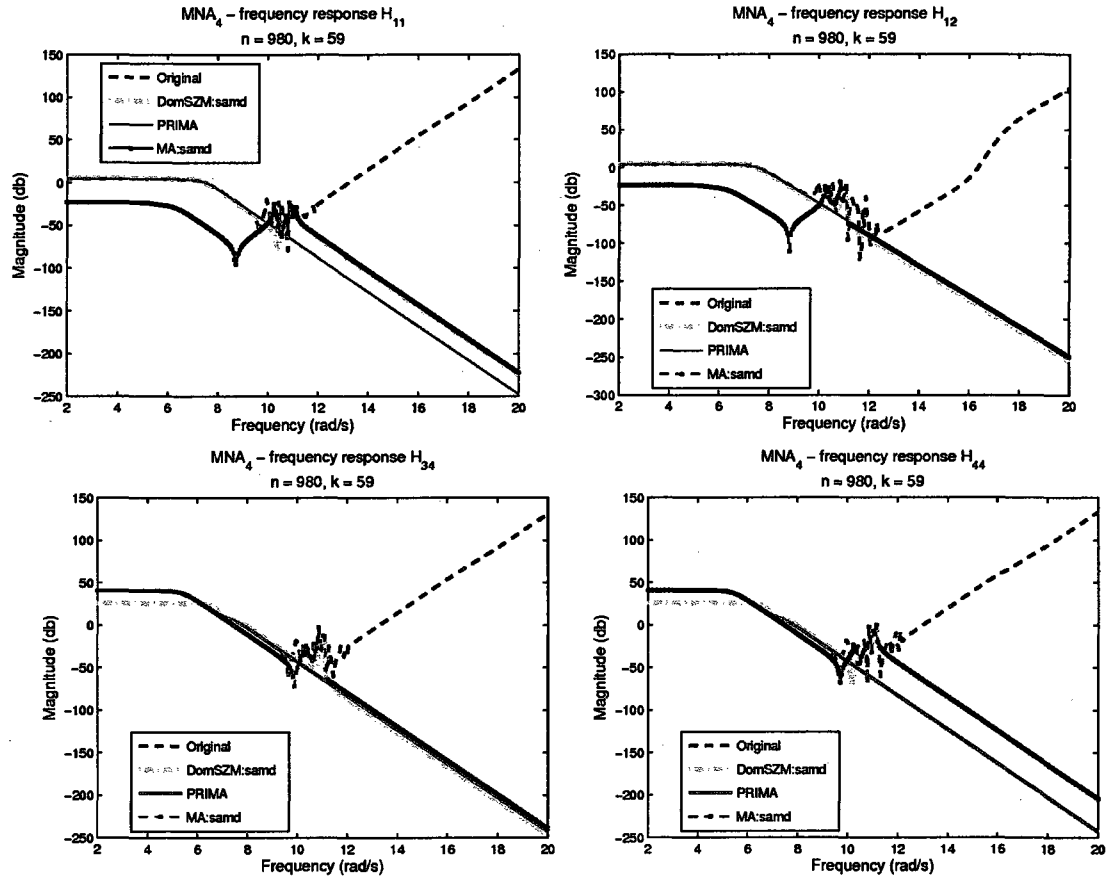


Figure 4.18 : MIMO MNA tline model. Individual transfer functions, original and reduced with dominant SZM, modal approximation and PRIMA.

4.5 Summary

Reduction for the voltage transfer function of transmission lines was addressed both from a SISO and MIMO perspective. The SISO method is novel, as reduction for SISO systems is generally performed with respect to the driving point admittance or impedance. The solution proposed here exploits the interpolation at dominant spectral zeros and insures that the reduced voltage transfer function is associated with a reduced model that can be synthesized with RLC circuit elements. Dominant SZM was also extended to handle MIMO systems, and the voltage transfer function reduction in this setting is comparable to the proposed SISO approach.

Chapter 5

Conclusions and future work

This thesis presents several new approaches for passive reduction of systems arising in circuit simulation. The foundation for preserving passivity is the spectral zero interpolation method [13, 14], which is improved as follows. A dominance criterion for selecting spectral zeros is proposed, and an iterative eigenvalue solver (SADPA [15, Chapter 3]) is tailored for automatic computation of dominant spectral zeros. SADPA also finds the associated invariant subspaces, which are used to construct the passivity preserving projection. The method reduces all passive circuits, including those not suitable for reduction with popular industry methods such as PRIMA. It is also shown how for certain descriptor systems from circuit simulation, dominant SZM and modal approximation can be combined efficiently with matching of Markov parameters of the fast subsystem. This improves the approximation and overcomes technical difficulties with dominant SZM when the system matrix $\mathbf{D}$ is 0. The problem of reducing transmission lines with respect to the voltage transfer function is also solved in this thesis. Based on the interpolatory nature of dominant SZM, a SISO reduction procedure is developed, which guarantees a realizable reduced model for the reduced voltage transfer function. Finally, the application of dominant SZM is extended for reduction of MIMO systems, based on the iterative solver SAMDP [15, Chapter 4].

The work in this thesis motivates further research in at least the following directions. The dominance criterion for spectral zero selection proposed here is by no

means unique. Other criteria such as $\mathcal{H}_2$ optimality [35] could be exploited and incorporated in SADPA, but the preservation of passivity is still an open question. In the context of Markov parameter matching for the fast subsystem associated with $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, methods for approximating the dominant eigenspace associated with the infinite eigenvalues of $(\mathbf{A}, \mathbf{E})$ are of interest. A step further would be passive reduction for $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with a singular $(\mathbf{A}, \mathbf{E})$ pencil, which often occurs in practice but is very little understood. Finally, to synthesize the reduced models as RLC circuits, a block structure preserving version of dominant SZM will be addressed, similar to the structure preserving SPRIM method proposed in [6].

Appendix A

Appendix

A.1 Proofs of theorems

Proofs for the theorems in Chapter 2 are included in this section.

Proof A.1 (Proof of Theorem 2.1) Recalling definition (1.4), we first identify the form of the system Σ_Z , whose zeros are the spectral zeros of Σ .

$$\begin{aligned} \mathbf{H}_Z(s) &= \mathbf{H}(s) + \mathbf{H}^*(-s) = \mathbf{C}_Z(s\mathbf{E}_Z - \mathbf{A}_Z)^{-1}\mathbf{B}_Z + \mathbf{D}_Z, \text{ where} \\ \mathbf{A}_Z &= \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -\mathbf{A}^* \end{pmatrix}, \mathbf{B}_Z = \begin{pmatrix} \mathbf{B} \\ -\mathbf{C}^* \end{pmatrix}, \mathbf{E}_Z = \begin{pmatrix} \mathbf{E} & 0 \\ 0 & \mathbf{E}^* \end{pmatrix} \\ \mathbf{C}_Z &= \begin{pmatrix} \mathbf{C} & \mathbf{B}^* \end{pmatrix}, \mathbf{D}_Z = \mathbf{D} + \mathbf{D}^*. \end{aligned} \tag{A.1}$$

The associated system is $\Sigma_Z = \left[\begin{array}{c|c} (\mathbf{A}_Z, \mathbf{E}_Z) & \mathbf{B}_Z \\ \hline \mathbf{C}_Z & \mathbf{D}_Z \end{array} \right]$. Using [15, Theorem 5.3.6], the

system with transfer function $\mathbf{G}(s) = \mathbf{H}_Z(s)^{-1}$ is $\Sigma_{\mathbf{G}} = \left[\begin{array}{c|c} (\mathbf{A}_h, \mathbf{E}_h) & \mathbf{B}_h \\ \hline \mathbf{C}_h & \Delta \end{array} \right]$, obtained from Σ_Z as follows:

$$\begin{aligned} \mathbf{A}_h &= \mathbf{A}_Z - \mathbf{B}_Z \mathbf{D}_Z^{-1} \mathbf{C}_Z^*, \quad \mathbf{B}_h = \mathbf{B}_Z \mathbf{D}_Z^{-1}, \quad \mathbf{E}_h = \mathbf{E}_Z \\ \mathbf{C}_h &= -\mathbf{D}_Z^{-1} \mathbf{C}_Z, \quad \Delta = \mathbf{D}_Z^{-1}. \end{aligned}$$

From basic operations on (A.1), the Hamiltonian system matrices in the form (2.2), (2.3) are obtained immediately.

Proof A.2 (Proof of Theorem 2.2) 1) Using the Schur complement of $-D-D^*$:

$$\begin{aligned}
& C_s(sE_s - A_s)^{-1}B_s + \Delta \\
&= -\Delta \begin{pmatrix} C^* \\ B \\ 0 \end{pmatrix}^* \begin{pmatrix} sE - A & 0 & -B \\ 0 & sE^* + A^* & C^* \\ -C & -B^* & -D - D^* \end{pmatrix}^{-1} \begin{pmatrix} B \\ -C^* \\ 0 \end{pmatrix} \Delta + \Delta \\
&= -\Delta \begin{pmatrix} C & B^* & 0 \end{pmatrix} \left(\begin{array}{c|c} T^{-1} & * \\ \hline * & * \end{array} \right) \begin{pmatrix} B \\ -C^* \\ 0 \end{pmatrix} \Delta + \Delta \\
&= -\Delta \begin{pmatrix} C & B^* \end{pmatrix} T^{-1} \begin{pmatrix} B \\ -C^* \end{pmatrix} \Delta + \Delta \\
&= C_h(sE_h - A_h)^{-1}B_h + \Delta = G(s), \text{ where} \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
T &= \begin{pmatrix} sE - A & 0 \\ 0 & sE^* + A^* \end{pmatrix} - \begin{pmatrix} -B \\ C^* \end{pmatrix} (-D - D^*)^{-1} \begin{pmatrix} -C & -B^* \end{pmatrix} \\
&= s \begin{pmatrix} E & 0 \\ 0 & E^* \end{pmatrix} - \begin{pmatrix} A - B\Delta C & -B\Delta B^* \\ C^*\Delta C & -A^* + C^*\Delta B^* \end{pmatrix} \\
\Delta &= (D + D^*)^{-1}.
\end{aligned}$$

2) Eigenvalues of $(\mathbf{A}_s, \mathbf{E}_s)$ are all $s \in \mathbb{C}$ such that $\det(s\mathbf{E}_s - \mathbf{A}_s) = 0 \Leftrightarrow$

$$\begin{aligned}
0 &= \det \left[\begin{pmatrix} s\mathbf{E} - \mathbf{A} & 0 & -\mathbf{B} \\ 0 & s\mathbf{E}^* + \mathbf{A}^* & \mathbf{C}^* \\ -\mathbf{C} & -\mathbf{B}^* & -\mathbf{D} - \mathbf{D}^* \end{pmatrix} \right] \Leftrightarrow \\
0 &= \det \left[\begin{pmatrix} s\mathbf{E} - \mathbf{A} & 0 \\ 0 & s\mathbf{E}^* + \mathbf{A}^* \end{pmatrix} \right] \\
&\quad \cdot \det \left[-\mathbf{D} - \mathbf{D}^* + \begin{pmatrix} \mathbf{C}^* \\ \mathbf{B} \end{pmatrix}^* \begin{pmatrix} s\mathbf{E} - \mathbf{A} & 0 \\ 0 & s\mathbf{E}^* + \mathbf{A}^* \end{pmatrix}^{-1} \begin{pmatrix} -\mathbf{B} \\ \mathbf{C}^* \end{pmatrix} \right] \Leftrightarrow \\
0 &= \det[(s\mathbf{E} - \mathbf{A})] \cdot \det[(-s\mathbf{E}^* - \mathbf{A}^*)] \\
&\quad \cdot \det \left[\begin{pmatrix} \mathbf{C}^* \\ \mathbf{B} \end{pmatrix}^* \begin{pmatrix} s\mathbf{E} - \mathbf{A} & 0 \\ 0 & s\mathbf{E}^* + \mathbf{A}^* \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{B} \\ -\mathbf{C}^* \end{pmatrix} + (\mathbf{D} + \mathbf{D}^*) \right] \Leftrightarrow \\
0 &= \underbrace{\det[(s\mathbf{E} - \mathbf{A})] \cdot \det[(-s\mathbf{E}^* - \mathbf{A}^*)]}_{\neq 0} \det[\mathbf{H}(s) + \mathbf{H}^*(-s)].
\end{aligned}$$

Pencil $(\mathbf{A}, \mathbf{E})$ is regular, thus any $s \in \mathbb{C}$ such that $\det[\mathbf{H}(s) + \mathbf{H}^*(-s)] = 0$ is a spectral zero of Σ , according to definition (1.4). They are the same as the eigenvalues of $(\mathbf{A}_h, \mathbf{E}_h)$, as evident from (A.2).

Proof A.3 (Proof of Theorem 2.3) Setting $\mathbf{D}_Z = 0$ in $\mathbf{H}_Z(s)$ in (A.1), we obtain the system Σ_Z whose zeros are the spectral zeros of $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$. Using [15, Theorem 5.3.5], the Hamiltonian system $\Sigma_h(\mathbf{E}_h, \mathbf{A}_h, \mathbf{B}_h, \mathbf{C}_h, 0)$ with transfer function $\mathbf{G}(s) =$

$\mathbf{H}_Z(s)^{-1}$ is obtained from Σ_Z as follows:

$$\begin{aligned} \mathbf{A}_h &= \begin{pmatrix} \mathbf{A}_Z & \mathbf{B}_Z \\ \mathbf{C}_Z^* & 0 \end{pmatrix}, \quad \mathbf{E}_h = \begin{pmatrix} \mathbf{E}_Z & 0 \\ 0 & 0 \end{pmatrix} \\ \mathbf{B}_h &= \begin{pmatrix} \mathbf{B}_Z \\ \mathbf{I} \end{pmatrix}, \quad \mathbf{C}_h = -\begin{pmatrix} \mathbf{C}_Z & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_h = 0. \end{aligned}$$

Proof A.4 (Proof of Lemma 2.1) Solving for $\mathbf{Z}_\delta = \delta^{-1}(\mathbf{C}\mathbf{X}_\delta + \mathbf{B}^*\mathbf{Y}_\delta)$ from the third equation in (2.17) and plugging it in the first and second, transforms the Hamiltonian eigenvalue problem into (A.3):

$$\left[\begin{pmatrix} \mathbf{A} & 0 \\ 0 & -\mathbf{A}^* \end{pmatrix} - \begin{pmatrix} \mathbf{B} \\ -\mathbf{C}^* \end{pmatrix} \delta^{-1} \begin{pmatrix} \mathbf{C} & \mathbf{B}^* \end{pmatrix} \right] \begin{pmatrix} \mathbf{X}_\delta \\ \mathbf{Y}_\delta \end{pmatrix} = \begin{pmatrix} \mathbf{E} & 0 \\ 0 & \mathbf{E}^* \end{pmatrix} \begin{pmatrix} \mathbf{X}_\delta \\ \mathbf{Y}_\delta \end{pmatrix} \Lambda_\delta. \quad (\text{A.3})$$

As $\delta \rightarrow \infty$, (A.3) approaches the generalized eigenvalue problem (A.4):

$$\begin{pmatrix} \mathbf{A} & 0 \\ 0 & -\mathbf{A}^* \end{pmatrix} \begin{pmatrix} \mathbf{X}_\delta \\ \mathbf{Y}_\delta \end{pmatrix} = \begin{pmatrix} \mathbf{E} & 0 \\ 0 & \mathbf{E}^* \end{pmatrix} \begin{pmatrix} \mathbf{X}_\delta \\ \mathbf{Y}_\delta \end{pmatrix} \Lambda_\delta, \quad (\text{A.4})$$

where $\Lambda_\delta = \text{diag}([\lambda_1, \dots, \lambda_n, -\lambda_1^*, \dots, -\lambda_n^*])$. Since $\delta \rightarrow \infty$, these points are close to the *poles* of the original system (the generalized eigenvalues of $(\mathbf{A}, \mathbf{E})$), together with their mirror images. This motivates projecting the original system with $\tilde{\mathbf{Y}}_\delta$ and $\tilde{\mathbf{X}}_\delta$. $\tilde{\mathbf{Y}}_\delta$ and $\tilde{\mathbf{X}}_\delta$ are passivity preserving projection matrices, because they interpolate $\lambda_1, \dots, \lambda_n$ and their mirror images $-\lambda_1^*, \dots, -\lambda_n^*$, thus satisfying the positive real lemma [13], [14], [3].

From (A.4) and partitioning $\Lambda_\delta = \text{diag}([\Lambda, -\Lambda^*])$, we obtain:

$$\mathbf{A}\tilde{\mathbf{X}}_\delta = \mathbf{E}\tilde{\mathbf{X}}_\delta\Lambda \quad (\text{A.5})$$

$$-\mathbf{A}^*\tilde{\mathbf{Y}}_\delta = \mathbf{E}^*\tilde{\mathbf{Y}}_\delta(-\Lambda^*) \Rightarrow \tilde{\mathbf{Y}}_\delta^*\mathbf{A} = \tilde{\mathbf{Y}}_\delta^*\mathbf{E}\Lambda \quad (\text{A.6})$$

From (A.5) and (A.6), as $\delta \rightarrow \infty$, the projecting matrices $\tilde{\mathbf{Y}}_\delta$ and $\tilde{\mathbf{X}}_\delta$ approach the left and right eigenvectors of $(\mathbf{A}, \mathbf{E})$ respectively, corresponding to eigenvalues Λ . Consequently, the poles of the reduced system (2.19), i.e., the generalized eigenvalues of $(\tilde{\mathbf{Y}}_\delta^*\mathbf{E}\tilde{\mathbf{X}}_\delta, \tilde{\mathbf{Y}}_\delta^*\mathbf{A}\tilde{\mathbf{X}}_\delta)$ are stable, located close to the dominant poles of $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$.

A.2 Reduction with the dominant spectral method

The procedure for passive reduction with dominant SZM is summarized next:

- 1) Given $\Sigma(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, construct the associated Hamiltonian system Σ_s described by (A.7), whose poles are the spectral zeros of Σ .

$$\begin{aligned} \mathbf{A}_s &= \begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & -\mathbf{A}^* & -\mathbf{C}^* \\ \mathbf{C} & \mathbf{B}^* & \mathbf{D} + \mathbf{D}^* \end{pmatrix}, \mathbf{E}_s = \begin{pmatrix} \mathbf{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \mathbf{B}_s = \begin{pmatrix} \mathbf{B} \\ -\mathbf{C}^* \\ \mathbf{0} \end{pmatrix} \Delta \\ \mathbf{C}_s &= -\Delta \begin{pmatrix} \mathbf{C} & \mathbf{B}^* & \mathbf{0} \end{pmatrix}, \mathbf{D}_s = (\mathbf{D} + \mathbf{D}^*)^{-1} \end{aligned} \quad (\text{A.7})$$

- 2) Solve the Hamiltonian eigenvalue problem

$(\Lambda, \mathbf{R}, \mathbf{L}) = \text{eig}(\mathbf{A}_s, \mathbf{E}_s)$, i.e., $\mathbf{A}_s\mathbf{R} = \mathbf{E}_s\mathbf{R}\Lambda$, $\mathbf{L}^*\mathbf{A}_s = \Lambda\mathbf{L}^*\mathbf{E}_s$. Eigenvalues $\Lambda = \text{diag}(s_1, \dots, s_n, -s_1^*, \dots, -s_n^*)$ are the spectral zeros of Σ , and $\mathbf{R} = [\mathbf{r}_1, \dots, \mathbf{r}_{2n}]$, $\mathbf{L} = [\mathbf{l}_1, \dots, \mathbf{l}_{2n}]$.

- 3) Compute residues R_j associated with the stable* spectral zeros s_j , $j = 1 \dots n$ as follows: $R_j = \gamma_j \beta_j$, $\gamma_j = \mathbf{C}_s \mathbf{r}_j (\mathbf{I}_j^* \mathbf{E}_s \mathbf{r}_j)^{-1}$, $\beta_j = \mathbf{I}_j^* \mathbf{B}_s$.
- 4) Sort spectral zeros descendingly according to dominance criterion $\frac{|R_j|}{|\operatorname{Re}(s_j)|}$ [15, Chapter 3], and reorder right eigenvectors $\mathbf{R}$ accordingly.
- 5) Retain the right eigenspace $\hat{\mathbf{R}} = [\mathbf{r}_1, \dots, \mathbf{r}_k] \in \mathbb{C}^{2n \times k}$, corresponding to the stable k most dominant spectral zeros.
- 6) Construct passivity projection matrices $\mathbf{V}$ and $\mathbf{W}$ from the rows of $\hat{\mathbf{R}}$: $\mathbf{V} = \hat{\mathbf{R}}_{[1:n, 1:k]}$, $\mathbf{W} = \hat{\mathbf{R}}_{[n+1:2n, 1:k]}$, and reduce Σ according to (1.3).

As explained in [13, 14] and Sect. 2.2.1, by projecting with (1.3), $\hat{\Sigma}$ interpolates the k most dominant spectral zeros of Σ , guaranteeing passivity and stability. For large-scale applications, a full solution to the eigenvalue problem in step 2), followed by the dominant sort 3)-4) is computationally unfeasible. Instead, the iterative solver SADPA (developed in [15, Chapter 3]) is applied with appropriate adaptations for spectral zero computation as described in Appendix A.3. The k most dominant spectral zeros and associated $2n \times k$ right eigenspace $\hat{\mathbf{R}}$ are obtained automatically.

A.3 Subspace accelerated dominant pole algorithm

The following pseudocode is extracted from [15, Chapter 3] and [11], with efficient modifications to automatically account for the four-fold symmetry $(\lambda, -\lambda^*, \lambda^*, -\lambda)$ of spectral zeros. In particular, as soon as a Hamiltonian eigenvalue (spectral zero) λ has converged, the right/left eigenvectors corresponding to $-\lambda^*$ are immediately deflated as well. It turns out that the right/left eigenvectors corresponding to $-\lambda^*$ need not

* $s \in \mathbb{C}$ is stable if $\operatorname{Re}(s) < 0$.

be solved for explicitly. Rather, due to the structure of the Hamiltonian matrices [7], [8], they can be written down directly from the already converged left/right eigenvectors for λ , as shown in steps 13-16 of Algorithm 1. As for modal approximation [11], [15, Chapter 3] deflation for λ^* and $-\lambda$ is automatically handled in Algorithm 3. To summarize, once the right/left eigenvectors corresponding to an eigenvalue λ have converged, the right/left eigenvectors corresponding to $-\lambda^*$, λ^* , $-\lambda$ are also readily available at no additional computational cost, and can be immediately deflated.

In Algorithm 2, the MATLAB `qz` routine is proposed for solving the small, projected eigenvalue problem in step 1. This reveals the right/left eigenvectors $\tilde{\mathbf{X}}, \tilde{\mathbf{V}}$ of the projected pencil directly, however they are neither orthogonal nor bi-G-orthogonal. Thus the normalization in step 3 is needed when computing the residues.

A modified Gram-Schmidt procedure (MGS) is used throughout for orthonormalization, following the implementation in [15, Algorithm 1.4]. For complete mathematical and algorithmic details of SADPA we refer to [15, Chapter 3] and [11].

Algorithm 1 $(\Lambda, R, L) = \text{SADPA}(\mathbf{E}_h, \mathbf{A}_h, \mathbf{B}_h, \mathbf{C}_h, s_1, \dots, p_{\max}, k_{\min}, k_{\max})$

Given: $(\mathbf{E}_h, \mathbf{A}_h, \mathbf{B}_h, \mathbf{C}_h)$, $\mathbf{E}_h \in \mathbb{C}^{2n \times 2n}$, $\mathbf{A}_h \in \mathbb{C}^{2n \times 2n}$, $\mathbf{B}_h \in \mathbb{C}^{2n \times 1}$, $\mathbf{C}_h \in \mathbb{C}^{1 \times 2n}$ an initial pole estimate s_1 and number of desired poles $p_{\max}$ (in the restarted version, $k_{\min}$ and $k_{\max}$ are also specified)

Output: Λ , the $p_{\max}$ most dominant eigenvalues and associated right, left eigenspaces R , L of $(\mathbf{A}_h, \mathbf{E}_h)$

```

1:  $k = 1$ ,  $p_{\text{found}} = 0$ ,  $\Lambda = []$ ,  $R = []$ ,  $L = []$ 
2: while  $p_{\text{found}} < p_{\max}$  do
3:   Solve for  $\mathbf{x}$  from  $(s_k \mathbf{E}_h - \mathbf{A}_h)\mathbf{x} = \mathbf{B}_h$ 
4:   Solve for  $\mathbf{v}$  from  $(s_k \mathbf{E}_h - \mathbf{A}_h)^* \mathbf{v} = \mathbf{C}_h^*$ 
5:    $\mathbf{x} = \text{MGS}(\mathbf{X}, \mathbf{x})$ ,  $\mathbf{X} = [\mathbf{X}, \mathbf{x}/\|\mathbf{x}\|]$ 
6:    $\mathbf{v} = \text{MGS}(\mathbf{V}, \mathbf{v})$ ,  $\mathbf{V} = [\mathbf{V}, \mathbf{v}/\|\mathbf{v}\|]$ 
7:   Compute  $\mathbf{G} = \mathbf{V}^* \mathbf{E}_h \mathbf{X}$  and  $\mathbf{T} = \mathbf{V}^* \mathbf{A}_h \mathbf{X}$ 
8:    $(\tilde{\Lambda}, \tilde{\mathbf{X}}, \tilde{\mathbf{V}}) = \text{DomSort}(\mathbf{T}, \mathbf{G}, \mathbf{X}, \mathbf{V}, \mathbf{B}_h, \mathbf{C}_h)$  ▷ Algorithm 2
9:   Compute dominant approximate eigentriplet  $(\hat{\lambda}_1, \hat{\mathbf{x}}_1, \hat{\mathbf{v}}_1)$ :
       
$$\hat{\lambda}_1 = \tilde{\lambda}_1, \hat{\mathbf{x}}_1 = (\mathbf{X} \tilde{\mathbf{x}}_1) / \|\mathbf{X} \tilde{\mathbf{x}}_1\|, \hat{\mathbf{v}}_1 = (\mathbf{V} \tilde{\mathbf{v}}_1) / \|\mathbf{V} \tilde{\mathbf{v}}_1\|$$

10:  if  $\|\mathbf{A}_h \hat{\mathbf{x}}_1 - \mathbf{E}_h \hat{\lambda}_1 \hat{\mathbf{x}}_1\| < \epsilon$  then
11:     $(\Lambda, R, L, \mathbf{X}, \mathbf{V}, \mathbf{B}_h, \mathbf{C}_h) = \text{Deflate}(\hat{\lambda}_1, \hat{\mathbf{x}}_1, \hat{\mathbf{v}}_1, \dots$ 
        $\Lambda, R, L, \mathbf{X} \tilde{\mathbf{X}}_{(:, 2:k)}, \mathbf{V} \tilde{\mathbf{V}}_{(:, 2:k)}, \dots$ 
        $\mathbf{E}_h, \mathbf{B}_h, \mathbf{C}_h)$  ▷ Algorithm 3
12:     $p_{\text{found}}++$ 
       ▷ Also find eigenvectors for the antistable spectral zero  $-\hat{\lambda}_1^*$  and deflate
13:     $\mathbf{x} = [ -\hat{\mathbf{v}}_{1(n+1:2n,:)}; \hat{\mathbf{v}}_{1(1:n,:)} ]$ 
14:     $\mathbf{v} = [ \hat{\mathbf{x}}_{1(n+1:2n,:)}; -\hat{\mathbf{x}}_{1(1:n,:)} ]$ 
15:     $(\Lambda, R, L, \mathbf{X}, \mathbf{V}, \mathbf{B}_h, \mathbf{C}_h) = \text{Deflate}(-\hat{\lambda}_1^*, \mathbf{x}, \mathbf{v}, \dots$ 
        $\Lambda, R, L, \mathbf{X}, \mathbf{V}, \mathbf{E}_h, \mathbf{B}_h, \mathbf{C}_h)$  ▷ Algorithm 3
16:     $p_{\text{found}}++$ 
17:     $\hat{\lambda}_1 = \tilde{\lambda}_2$ 
18:  else if  $\text{ncols}(\tilde{\mathbf{X}}) > k_{\max}$  then ▷ Possible restart
    ▷ Retain first  $k_{\min}$  most dominant approximate eigenvectors and re-orthonormalize
19:     $\mathbf{X} = \text{MGS}(\mathbf{X} \tilde{\mathbf{X}}_{(:, 1:k_{\min})})$  ▷ orthonormalize all columns sequentially
20:     $\mathbf{V} = \text{MGS}(\mathbf{V} \tilde{\mathbf{V}}_{(:, 1:k_{\min})})$ 
21:  end if
22:  Increment  $k = k + 1$ 
23:  Select new most dominant pole estimate  $s_k = \tilde{\lambda}_1$ 
24: end while

```

Algorithm 2 $(\tilde{\Lambda}, \tilde{\mathbf{X}}, \tilde{\mathbf{V}}) = \text{DomSort}(\mathbf{T}, \mathbf{G}, \mathbf{X}, \mathbf{V}, \mathbf{B}_h, \mathbf{C}_h)$

Given: $(\mathbf{T}, \mathbf{G}), \mathbf{X}, \mathbf{V}, \mathbf{B}_h, \mathbf{C}_h$

Output: $(\tilde{\Lambda}, \tilde{\mathbf{X}}, \tilde{\mathbf{V}})$, k dominant approximate eigenvalues and associated right, left eigenvectors of $(\mathbf{T}, \mathbf{G})$, sorted such that $\tilde{\lambda}_1$ is most dominant

1: $(\mathbf{A}\mathbf{A}, \mathbf{B}\mathbf{B}, \mathbf{Q}, \mathbf{Z}, \tilde{\mathbf{X}}, \tilde{\mathbf{V}}) = \text{QZ}(\mathbf{T}, \mathbf{G})$

2: $\tilde{\Lambda} = \text{diag}(\mathbf{A}\mathbf{A}) ./ \text{diag}(\mathbf{B}\mathbf{B})$ and $|\tilde{\lambda}_i| \neq \infty, i = 1 \dots k$

3: $R_i = \frac{[\mathbf{C}_h \tilde{\mathbf{x}}_i][\tilde{\mathbf{v}}_i^* \mathbf{B}_h]}{\tilde{\mathbf{v}}_i^* \mathbf{G} \tilde{\mathbf{x}}_i}$

▷ Compute residues

4: Sort $(\tilde{\Lambda}, \tilde{\mathbf{X}}, \tilde{\mathbf{V}})$ in decreasing $|R_i|/|Re(\tilde{\lambda}_i)|$ order

Algorithm 3 $(\mathbf{\Lambda}, \mathbf{R}, \mathbf{L}, \mathbf{X}, \mathbf{V}, \mathbf{B}_h, \mathbf{C}_h) = \text{Deflate}(\hat{\lambda}, \hat{\mathbf{x}}, \hat{\mathbf{v}}, \dots, \mathbf{\Lambda}, \mathbf{R}, \mathbf{L}, \hat{\mathbf{X}}, \hat{\mathbf{V}}, \mathbf{E}_h, \mathbf{B}_h, \mathbf{C}_h)$

Given: $(\hat{\lambda}, \hat{\mathbf{x}}, \hat{\mathbf{v}})$: the newly converged most dominant eigentriplet, $(\mathbf{\Lambda}, \mathbf{R}, \mathbf{L})$: the dominant eigentriplets already found correctly, $\hat{\mathbf{X}}, \hat{\mathbf{V}}$: the approximate right/left eigenvectors not yet checked for convergence, $\mathbf{E}_h, \mathbf{B}_h, \mathbf{C}_h$

Output: $(\mathbf{\Lambda}, \mathbf{R}, \mathbf{L})$: updated converged eigentriplets, $\mathbf{X}, \mathbf{V}$: deflated approximate eigenspaces, $\mathbf{B}_h, \mathbf{C}_h$: deflated matrices

1: $\mathbf{\Lambda} = [\mathbf{\Lambda}, \hat{\lambda}]$

2: $\hat{\mathbf{r}} = \hat{\mathbf{x}} / (\hat{\mathbf{v}}^* \mathbf{E}_h \hat{\mathbf{x}})$

▷ for keeping converged eigenvectors bi- $\mathbf{E}$ -orthogonal

3: $\hat{\mathbf{l}} = \hat{\mathbf{v}}$

4: $\mathbf{R} = [\mathbf{R}, \hat{\mathbf{r}}], \mathbf{L} = [\mathbf{L}, \hat{\mathbf{l}}]$

5: Deflate $\mathbf{B}_h = \mathbf{B}_h - \mathbf{E}_h \hat{\mathbf{r}} (\hat{\mathbf{l}}^* \mathbf{B}_h)$

6: Deflate $\mathbf{C}_h = \mathbf{C}_h - (\mathbf{C}_h \hat{\mathbf{r}}) \hat{\mathbf{l}}^* \mathbf{E}_h$

7: if $\text{imag}(\hat{\lambda}) \neq 0$ then

▷ also deflate complex conjugate

8: $\mathbf{\Lambda} = [\mathbf{\Lambda}, \hat{\lambda}^*]$

9: $\hat{\mathbf{r}} = \hat{\mathbf{r}}^*, \hat{\mathbf{l}} = \hat{\mathbf{l}}^*$

10: $\mathbf{R} = [\mathbf{R}, \hat{\mathbf{r}}], \mathbf{L} = [\mathbf{L}, \hat{\mathbf{l}}]$

11: Deflate $\mathbf{B}_h = \mathbf{B}_h - \mathbf{E}_h \hat{\mathbf{r}} (\hat{\mathbf{l}}^* \mathbf{B}_h)$

12: Deflate $\mathbf{C}_h = \mathbf{C}_h - (\mathbf{C}_h \hat{\mathbf{r}}) \hat{\mathbf{l}}^* \mathbf{E}_h$

13: end if

14: $\mathbf{X} = \mathbf{Y} = []$

15: for $j = 1 \dots \# \text{cols}(\hat{\mathbf{X}})$ do

16: $\mathbf{X} = \text{Expand}(\mathbf{X}, \mathbf{R}, \mathbf{L}, \mathbf{E}_h, \hat{\mathbf{x}}_j)$

▷ Algorithm 4

17: $\mathbf{V} = \text{Expand}(\mathbf{V}, \mathbf{R}, \mathbf{L}, \mathbf{E}_h^*, \hat{\mathbf{v}}_j)$

▷ Algorithm 4

18: end for

Algorithm 4 $\mathbf{X} = \text{Expand}(\mathbf{X}, \mathbf{R}, \mathbf{L}, \mathbf{E}_h, \hat{\mathbf{x}})$

Given: $\mathbf{X} \in \mathbb{C}^{2n \times k}$ such that $\mathbf{X}\mathbf{X}^* = \mathbf{I}$, $(\mathbf{R}, \mathbf{L}) \in \mathbb{C}^{2n \times p}$: the correctly found right/left eigenvectors such that: $\mathbf{L}^* \mathbf{E}_h \mathbf{R}$ is diagonal and $\mathbf{L}^* \mathbf{E}_h \mathbf{X} = \mathbf{0}$, $\hat{\mathbf{x}}$: approximate eigenvector not yet checked for convergence, $\mathbf{E}_h$

Output: $\mathbf{X} \in \mathbb{C}^{2n \times (k+1)}$ expanded such that $\mathbf{X}\mathbf{X}^* = \mathbf{I}$

▷ Project $\mathbf{x}$ onto $\text{ColSpan}(\mathbf{R})$ along $\text{ColSpan}(\mathbf{L})$

$$\mathbf{x}_{k+1} = \prod_{j=1}^p \left(\mathbf{I} - \frac{\mathbf{r}_j \mathbf{l}_j^* \mathbf{E}_h}{\mathbf{l}_j^* \mathbf{E}_h \mathbf{r}_j} \right) \hat{\mathbf{x}}$$

$$\mathbf{x} = \text{MGS}(\mathbf{X}, \mathbf{x}_{k+1})$$

$$\mathbf{X} = [\mathbf{X}, \mathbf{x} / \|\mathbf{x}\|]$$

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