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# **Dynamic Analysis of Stacked Rigid Blocks**

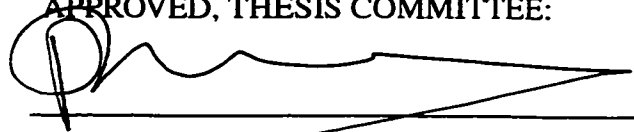
by

**Panayiotis C. Roussis**

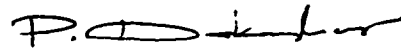
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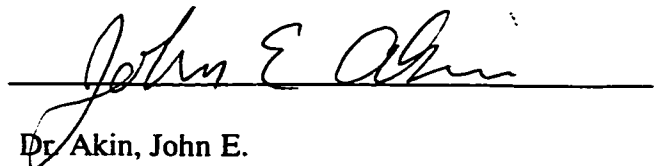
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## **Abstract**

### **Dynamic Analysis of Stacked Rigid Blocks**

**by**

**Panayiotis C. Roussis**

The dynamic behavior of structures of two stacked rigid blocks subjected to ground excitation is examined. Assuming no sliding, the rocking response of the system standing free on a rigid foundation is investigated. The analytical formulation of this nonlinear problem is quite challenging. Its complexity is associated with the continuous transition from one regime of motion to another, each one being governed by a different set of highly nonlinear equations. The behavior is described in terms of the four possible modes of response and impact between the two blocks, or the first block and the ground. The exact (nonlinear) equations governing the rocking response of the system to horizontal and vertical ground acceleration are derived for each mode. Furthermore, a comprehensive model governing impact is derived using classical impact theory. Finally, a computer program is developed to approximate the response of the system under arbitrary base excitation.

## **Acknowledgments**

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## **CHAPTER 1: Introduction**

### **1.1 Motivation**

It is common practice in structural engineering when dealing with structures under dynamic excitation, to assume deformable continuum behavior. For a number of structural systems, however, rigid body motion may well be a significant behavior mechanism. In fact, the seismic behavior of block-like structures standing free on their foundation, such as ancient monuments, petroleum storage tanks, water towers, nuclear reactors, concrete radiation shields, computer-type equipment, and priceless artifacts, has been the object of numerous researches for over a century.

Housner's landmark study (1963) has provided the basic understanding on the rocking response of a rigid block and sparked modern scientific interest. Recent years have seen a wealth of research papers on block-like structures, mainly devoted to the dynamics of a single block. The dynamics of a such apparently simple system has proved quite complicated and difficult to treat analytically due to its nonlinear nature.

As opposed to the case of a single rigid block, the dynamic behavior of multi-block structures has not been to date adequately studied. Even for the simplest case of multi-block structures, i.e., two-block assemblies, the rocking problem becomes very complex. Such configuration, in which one block is placed over the other, can be thought of as the model of ancient Greek and Roman type structures composed of large heavy members that simply lie on top of each other, or a piece of machinery or statue placed on top of a block-like base. This investigation was motivated by the concern of minimizing the damage of such structural systems during major earthquakes. In this regard, it is particularly important to

understand the dynamic behavior of these systems in order to check and possibly improve their durability.

### **1.2 Statement of the Problem**

This study deals with the dynamic behavior of structures consisting of two rigid blocks that simply lie on top of each other. Under the assumptions of rigid foundation, large friction to prevent sliding, and point contact during a perfectly plastic impact, the only possible response mechanism under base excitation is rocking about the corners of the blocks. In this regard, the most probable failure mechanism is the loss of equilibrium of the whole structure or part of it.

The analytical formulation of this nonlinear problem is quite challenging. Its complexity is associated with the continuous transition from one regime of motion to another, each one being governed by a different set of highly nonlinear equations. The primary contribution of this work is (a) the derivation of the exact (nonlinear) equations of motion for the system considered undergoing base excitation, (b) the treatment of the impact problem by deriving expressions for the post-impact angular velocities, and (c) the development of a comprehensive computer program for calculating the response of the system to base excitation.

### **1.3 Organization of the Work**

A review of the literature on the dynamic behavior of rigid blocks is presented in Chapter 2, where a brief summary of recent investigations is given. For details of the earlier work, the reader is referred to Ishiyama (1980).

Chapter 3 addresses the problem of a single rigid block. It serves as an introduction to the

analysis of the two-block system studied extensively in Chapters 5 and 6. The analytical models commonly used to study each mode of response are presented. Models include those for the slide, rock, slide-rock and free-flight modes.

In Chapter 4, the dynamic behavior of rocking systems consisting of two blocks, one placed on the top of the other, is examined. In particular, all different modes of response are classified and criteria for initiation of motion are derived. For each mode of response, the exact nonlinear equations of motion are derived and transition criteria between modes, in the absence of impact, are outlined.

In Chapter 5, the impact problem is discussed. Impact plays an important role in the dynamic analysis of the system, since impact is always accompanied by a mode transition. Using the principle of conservation of angular momentum, expressions for the post-impact angular velocities are derived.

Chapter 6 outlines a computer program developed for calculating the response of a two stacked rigid block system to base excitation. The program solves for the response under general conditions, considering all four modes of response, large angles of rotation, impact, and arbitrary base excitation. The full nonlinear equations of motion are integrated for each mode using a fourth-order Runge-Kutta integration scheme, and any potential transition between modes is automatically accounted for. To illustrate the practical application of the program, the response of the system in free vibration and under earthquake excitation has been computed for specific system parameters.

Finally, the summary and conclusions of this study are presented in Chapter 7.

## CHAPTER 2: Literature Review

In almost every destructive earthquake in the Eastern Mediterranean and Middle Eastern regions, free standing columns of Greek and Roman monuments and statues have survived undamaged in earthquakes that have caused spectacular destruction around them. In some cities in India, free-standing stone columns that supported heavy statues remained standing although at the end of the earthquake they were surrounded by heaps of debris that had been seemingly more stable structures. Tall, slender stone pillars in graveyards have survived strong ground motion whereas box-like electric power transformers have rocked and overturned. During the Chilean earthquakes of 1960, several golf-ball-on-a-tee type of elevated water tanks survived the ground shaking whereas much more stable-appearing reinforced concrete elevated water tanks were severely damaged. It is clear that, the concept of representing the effect of an earthquake by a certain *static* lateral force (percent *g* design) may be quite misleading. In order to explain such anomalous behavior, the above-mentioned analysis of overturning of rigid blocks, proposed several decades ago, are inadequate, and it becomes necessary to consider the *dynamics* of the system.

Researchers have been interested in the rocking response of rigid structures as early as 1881. Before the advent of strong motion accelerographs, techniques were proposed for estimating the intensity of ground shaking and predominant period associated with earthquakes based on observed overthrow of tombstones, monuments and statues. Due to the complexity of the problem, many relied on experimental results and basic analytic solution for simple ground motions. Ishiyama (1980) provides a comprehensive review of this early work, much of which is in Japanese. More recently, emphasis has been placed on



evaluating the performance of rigid structures during an earthquake for safety requirements. A brief summary of these recent investigations is given below; for details of the earlier work, the reader is referred to Ishiyama (1980).

Housner's landmark study (1963) has provided the basic understanding on the rocking response of a rigid block and sparked modern scientific interest. Motivated by the observations of damage to water tanks in the 1960 Chilean earthquakes, Housner was the first to investigate systematically the dynamics of a rigid block on a rigid horizontal base undergoing horizontal motion. Using the linearized equation of motion and a simple impact relation, free rocking (vibration) was first investigated. Representing the ground acceleration as a rectangular pulse and as a half-cycle sine-wave pulse, equations were derived for the minimum accelerations (which depend on the duration of the pulse) required to overturn a block. The approach employed in obtaining these equations is valid only for a single pulse excitations which, if large enough, would overturn the block without rocking and the associated impacts. Smaller accelerations may set the block to rocking but will not overturn it. It is, however, possible to overturn the block with smaller accelerations if a number of pulses act successively, a characteristic of ground accelerations during earthquakes. Using an energy approach and idealizing the ground motion as a white noise, Housner presented an approximate analysis of the dynamics of rigid blocks subjected to such excitations. From these results he demonstrated that there is a scale effect which makes the larger of two geometrically similar blocks more stable than the smaller block. Moreover, the robustness of a tall, slender block subjected to earthquake motion is much greater than would be inferred from a stability analysis to a static horizontal force, employed to represent earthquake effects in the simple analysis mentioned above.

In 1980, Aslam et al. examined the response of a rigid block subjected to horizontal and vertical ground accelerations, with the option of elastic tie-down rods and the assumption of no sliding. The full nonlinear equation of motion was solved numerically using a predictor-corrector algorithm for free vibration, harmonic ground motions and recorded accelerograms. Results were verified with experiments on a shaking table using a concrete block. Free and forced vibration tests compared well with numerical results. However, agreement with recorded earthquakes was poor. In addition, the experimental results for recorded earthquakes were not reproducible over repeated tests using the same accelerogram. This indicated the sensitivity of the response to the foundation conditions during impact.

In the same year, a probabilistic approach to the problem of rocking of rigid blocks was pursued by Yim et al. They studied the rocking behavior of rigid blocks subject to horizontal and vertical accelerations, assuming no sliding. The nonlinear equation of motion was solved numerically using a fourth-order Runge-Kutta integration scheme for a series of 20 artificial earthquakes. A study was conducted to determine the effect of various system parameters on the observed response. In addition, cumulative distribution functions for overturning were generated for a range of system parameters. It was shown that a probabilistic estimate of the intensity of ground shaking could be obtained from the observed effect of tombstones and monuments, provided sufficient data were available.

In 1982, Ishiyama analyzed impact from a rock, slide-rock and free-flight mode. Planar impact was assumed, along with non-zero coefficient of restitution and non-infinite coefficient of friction. The analysis was broken down into two phases. Initially, the resultant

impulses  $Q_x$  and  $Q_y$  were assumed to occur at some distance  $\xi$  measured from the center of mass. Relating the pre- and post-impact  $y$  velocities at this point through the coefficient of restitution,  $Q_y$  was evaluated. To determine  $Q_x$ , a new parameter, the “tangent coefficient of restitution” ( $-1 \leq e_t \leq 1$ ) was introduced which related the pre- and post-impact tangent velocities at the point of application of the impulse. With an assumed  $e$  and  $e_t$ , the resulting post-impact velocities were evaluated. If the post-impact solution violated the velocity constraint, a second series of impulses was applied in order to guarantee a post-impact rock or slide-rock mode.

A study on the dynamic behavior of a rocking rigid block supported by flexible foundations which permit up-lift, was performed by Psycharis and Jennings in 1983. They considered two foundation models, namely the Winkler model and a two-spring foundation, and established an equivalence between the two models, so that the equivalent two-spring model can be used instead of the Winkler foundation. Solutions of the linearized equations of motion were developed and demonstrated that lift-off results in a softer vibrating system which behaves non-linearly overall, although the response is composed of a sequence of linear responses. Since the foundation models considered cannot dissipate energy, in order to account for the energy dissipation from impact - which is expected to occur in actual situations where lift-off happens - a simplified impact mechanism was introduced.

In 1984, Spanos and Koh investigated the rocking response of a rigid block subject to harmonic ground motion, assuming no sliding. The linear and nonlinear equations of motion were solved numerically assuming zero initial conditions to identify likely steady-state modes of response. A number of modes were identified, including a fundamental mode at

the frequency of excitation, a  $1/3$  subharmonic and an unsymmetric (i.e., non-zero mean) mode at the fundamental frequency. Using the linearized equation of motion, a general piecewise-exact solution for steady-state response was then developed for the identified modes. Using the solution, steady-state amplitudes were obtained, and stability of the system was investigated. These results were also compared to an approximate solution obtained using the method of equivalent linearization.

In 1986, Allen et al., studied the dynamic behavior of an assembly of two-dimensional rigid prisms. Their model was related to a precast concrete building system, employing slender columns with negligible beam-column moment resistance, that would behave as an assembly of constrained rigid bodies under earthquake excitation. Linearized time histories were presented for one and two degree-of-freedom systems. As expected, the response modes of the systems were poorly conditioned; small changes in input or geometry can create large changes in system response.

In 1986, Koh et al., studied the behavior of a rigid block rocking on a flexible foundation. Modulated white noise was used as a model of horizontal acceleration of the foundation. The statistics of the rocking response were found by an analytical procedure which involved a combination of static condensation and stochastic linearization. The analytical solution was computationally efficient and compared well with pertinent data obtained by numerical simulations.

In 1989, Tso and Wong investigated the rocking response both analytically and experimentally. Their study aimed to determine the effect of various system parameters on the steady-state response. Experiments were conducted to verify the analysis. Peak steady-

state amplitudes for the fundamental mode were recorded and found to be in good agreement with the theory. In addition, a  $1/3$  subharmonic was recorded experimentally. Stability of the steady-state solutions was also investigated numerically.

In 1990, Hogan, who adopted the model, the analysis, and the response classification of Spanos and Koh (1984), performed a complete investigation of the existence and stability of single-impact subharmonic responses  $(1, n)$  (with  $n \geq 1$ ), as a function of the restitution coefficient  $\beta$ . Hogan verified the existence of motions, characterized by a period increasing with the amplitude of the harmonic excitation, until the response becomes aperiodic or more probably chaotic. Due to the high sensitivity of the response to the initial conditions, he determined the domains of attraction for four subharmonic orbits  $(1, n)$ : the behavior of the system was in some ways unpredictable.

In 1990, Psycharis presented an analysis on the dynamic behavior of systems consisting of two blocks. Assuming friction large enough to prevent sliding, all possible rocking modes were classified depending on the relative position of the blocks. Furthermore, assuming small angles of rotation, approximate (linearized) equations of motion were derived. The response of the system to horizontal earthquake excitation was calculated by time step integration.

In 1990, Sinopoli approached the impact problem by adopting a unilateral constraint, 'kinematic approach'. Her study assigned a pure sliding mode after impact for short blocks as having the aspect ratio less than  $\sqrt{2}$  for frictionless surfaces and less than  $(\sqrt{2})/2$  for very rough surfaces. She analyzed the dynamic stability with respect to the

overturning of a multi-block column excited by a sine wave ground motion.

One year later, Yim and Lin investigated the influence of nonlinearities associated with impact on the behavior of free-standing rigid objects subjected to horizontal base excitations. In addition to the periodic and overturning responses, they discovered two new types of steady state responses, quasi-periodic, and chaotic. An analytical technique based on the Melnikov function was derived for the prediction of the existence of chaotic response. The accuracy of the method was assessed by numerical results. The relationship between the Melnikov analysis developed in this study and the stability analysis developed by Spanos and Koh (1984) was also examined.

In 1991, Giannini and Masiani tackled the problem of the dynamic response of a rigid block oscillator to a Gaussian white noise excitation process. In this study, the equation of motion was linearized with the assumption of small displacements. The energy dissipation due to the inelastic impact was modeled as an impulsive process with arrivals occurring at displacement equal zero. The solution of the associated Fokker-Plank equation was obtained with two techniques of approximation: Gaussian closure and non-Gaussian closure. The first one gave a good approximation of the transient and the stationary approach when the intensity of the excitation was low, but it overestimated the response for stronger excitations. Non-Gaussian closure better approximated the stationary intensity.

In 1991, Shenton and Jones presented a general, two-dimensional formulation for the response of free-standing rigid bodies to base excitation. The formulation assumed rigid body, rigid foundation and Coulomb friction. The behavior was described in terms of the five possible modes of response, namely rest, slide, rock, slide-rock and free-flight and the

impact between the body and the foundation. A model governing impact from a rock, slide-rock and free-flight mode was derived using classical impact theory. This model assumed a point-impact, nonzero coefficient of restitution and finite-value of friction. An approximate closed-form solution was developed for a single-mode-steady-state-slide-rock response resulting from a harmonic ground acceleration, using the method of slowly varying parameters. This approximate solution is valid for a rectangular block undergoing small angles of rotation at the frequency of ground motion. Impact was assumed to be perfectly plastic and frictional impulses were included. Periodic solutions were found to exist in general only for relatively high amplitudes of ground acceleration and friction less than the inverse aspect ratio of the block. Results showed that the rock component of the response was sensitive to changes in aspect ratio and friction and insensitive to changes in ground acceleration. The slide components of the response was approximately equal to the amplitude of ground displacement and was insensitive to changes in friction and aspect ratio. Their results compared favorably to those obtained by numerical integration.

In 1992, Augusti and Sinopoli presented a formulation of dynamics and impact problem for a single rigid body freely supported on rigid foundation. A review was presented on the numerous researchers performed on this subject and the results achieved.

In the same year, Wang and Mason derived solutions for frictional planar rigid-body collisions, using Routh's impact process diagrams, for both Newtonian and Poisson restitution. They classified the possible modes of impact, and derived analytical expressions for impulses and motions of the bodies. The solution for the impact problem is applicable for impact between two rigid bodies that are in contact with each other at only a single distinct

point.

An investigation by Dimentberg et al. in 1993, addressed the toppling failure of a free standing and an anchored rigid block due to horizontal and vertical base excitations. They obtained expressions for the statistical properties and probability distribution of the random toppling time. The excitations were idealized as white noises. It was found that the correlation between the vertical and horizontal base excitations did not affect the averaged rocking motion; however, the presence of vertical excitation about one-half the level of horizontal excitation may increase the probability of toppling by as much as 30-40%. They, also, concluded that larger blocks are more stable than the smaller ones of the same geometrical proportion.

In 1995, Allen and Duan examined the effects of linearizing on rocking blocks. For free vibration, they found that the difference between the approximate and the exact quarter period increases as a block becomes less slender. The greatest difference was about 2%. Parameter studies between surviving and toppling demonstrated that models were least sensitive to the slenderness angle and the peak amplitude, more sensitive to the rebound velocity coefficient, and most sensitive to the block diagonal. This sensitivity was generally reduced when using the nonlinear model. Although meaningful results could be obtained with the linear model, they concluded that the nonlinear model provides more robust results, even for slender blocks.

In the same year, Cai et al. examined the toppling of a rigid body under random excitation, by modeling the base accelerations more realistically as evolutionary processes with broad-band spectra. A version of quasiconservative averaging was applied to account for



the nonstationary and nonwhite characteristics of the excitations. The total energy of the rocking block was approximated as a Markov process. The reliability of the system, namely the probability that toppling has not occurred up to a given time, was calculated by using the numerical procedure of path integration.

More recently, Lin and Yim (1996) examined the rocking behavior of slender rigid objects subjected to periodic excitations with and without noise disturbance for a better understanding of their response, stability and sensitivity. Taking into account the presence of noise, a mean-square criterion for the possible chaotic domain based on the stochastic Melnikov process has been derived. It was found that the presence of noise enlarged the chaotic domain. Relationships between chaotic and overturning responses have been analytically examined. Numerical results indicated that even with the presence of very weak noise the chaotic response trajectory will eventually be driven to overturning, thus weak stability of deterministic chaotic rocking response was indicated. Numerical results also confirmed that 'noisy and chaotic' response was a possible intermediate state between bounded (periodic) response and overturning. In the companion paper, Lin and Yim also examined, from a probabilistic perspective, the rocking responses of fully nonlinear rocking systems subjected to combined deterministic and stochastic excitations. They concluded that the Fokker-Planck equation provided a probabilistic description of the rocking response behaviors to combined deterministic and stochastic excitations. In addition, the resulting probability density functions provided global information about the response behaviors. The long-term rocking response behavior was inherently unstable when stochastic excitation was present. They also performed sensitivity studies of the response to

relative randomness parameter  $\gamma$ , system slenderness ratio  $r$ , velocity restitution coefficient  $e$ , as well as uncertainties in initial conditions.

In 1996, Gianini and Masiani used a random vibration approach to study the response of a slender rigid block to seismic excitation. The problem of rocking with impulsive dissipation of slender rigid blocks under horizontal base acceleration was solved by means of a non-Gaussian closure method assuming a Laplace marginal distribution for the displacement and a Gaussian marginal distribution for the velocity. Two cases were considered to simulate seismic shaking: a Gaussian stationary white noise, and a filtered Gaussian white noise excitation of Kanai-Tajimi type. The analytical solutions were obtained solving initial value problems with ordinary differential equations. The mean upcrossing rates and the response spectra in terms of displacement were evaluated. The reliability of the solutions derived was assessed by comparing them with Monte Carlo simulations.

In 1997, Sinopoli investigated the dynamics of a rigid body simply supported on a moving rigid ground in the presence of dry friction. Rigid body kinematics, in terms of generalized coordinates, and features of contact were discussed. A variational formulation was adopted to describe the dynamics of the system. A geometric method of the variational formulation then proposed in the configuration space, which allowed the acceleration to be determined for any kind of situation without any direct evaluation of the contact forces.

In 1998, Pompei et al. studied the dynamics of a rigid block subjected to horizontal ground motion, the objective being to formulate criteria that separate the stucked and slipped modes of the motion. In this study, they investigated the influence of three parametric quantities, namely, the static friction coefficient  $\mu_s$ , the aspect ratio  $b/h$  of the block, and

the normalized ground acceleration  $a/g$ . It was shown that, in some cases, sliding response should be taken into account in the study of the body rotation.

## CHAPTER 3: Dynamic Response of a Rigid Block

### 3.1 Preliminaries

Consider a symmetric rigid body of mass  $m$  and centroid mass moment of inertia  $I$ , supported on a horizontal plane rigid ground in its static equilibrium configuration; see Figure 3.1. Let  $H$  and  $B$  be the height and the base width, respectively, of the block, and  $\mu_s$  and  $\mu_k$  the static and kinetic friction coefficients, respectively, for the materials in contact. The distance from either base corner to the center of mass is denoted by  $r$ , and the angle measured between  $r$  and the vertical when the body is at rest is denoted by  $\theta_c$ .

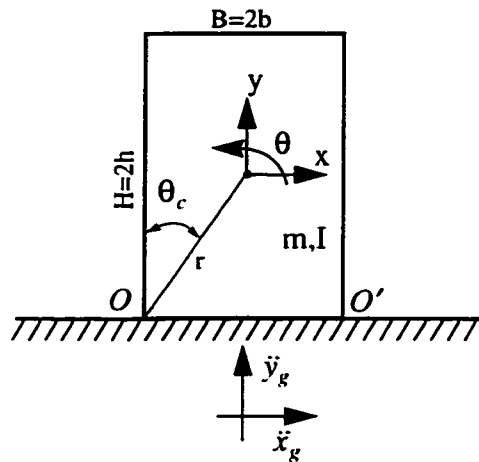


Fig. 3.1 A Rigid Block at Rest

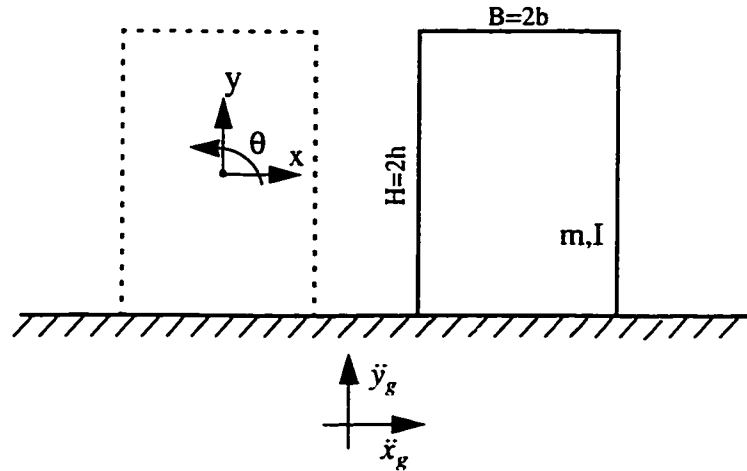
When subjected to a base acceleration with horizontal and vertical components  $\ddot{x}_g$  and  $\ddot{y}_g$  respectively, the block may move rigidly with the ground (rest mode), or be set into relative motion. In the latter case, the block can be set into slide, rock, slide-rock, or free-flight mode. The five possible modes of response are illustrated schematically in Figures 3.1 through 3.5. Horizontal and vertical displacements of the mass center relative to the foun-

ation are denoted by  $x(t)$  and  $y(t)$ , respectively. The tilting of the block from the vertical is measured by the angle  $\theta(t)$ , positive in the counter-clockwise sense.

### 3.2 Analytical Models

#### *Slide Mode*

This mode describes the relative horizontal translation of the block with respect to the ground. It is characterized by a normal reaction force greater than zero ( $f_y > 0$ ), a friction force which is a function of the normal force and the velocity of the center of mass, and zero rotation ( $\theta(t) = 0$ ).



**Fig. 3.2** Slide Mode of a Rigid Block

Assuming that the body is initially at rest, a slide mode (Figure 3.2) is initiated once the inertial load of the mass exceeds the resistance provided by friction, namely,

$$|m\ddot{x}_g| > \mu_s(m\ddot{y}_g + mg)$$

or

$$|\ddot{x}_g| > \mu_s(\ddot{y}_g + g) \quad (3.1)$$

In the special case that  $\ddot{y}_g = 0$ , Equation (3.1) reduces to

$$|\ddot{x}_g| > \mu_s g$$

Once the mass is sliding, the governing equation of motion is given by

$$m(\ddot{x} + \ddot{x}_g) = -S(\dot{x})\mu_k m(\ddot{y}_g + g), \quad (3.2)$$

where  $S(\dot{x})$  denotes the signum function in  $\dot{x}$ , defined by

$$S(\dot{x}) = \begin{cases} 1 & \dot{x} > 0 \\ -1 & \dot{x} < 0 \end{cases} \quad (3.3)$$

Sliding lasts until the relative velocity of the mass equals zero ( $\dot{x}=0$ ). At that time, the body momentarily comes to rest relative to the foundation and Equation (3.1) is used to determine whether it remains at rest or continues to slide. Depending on the coefficient of friction and the characteristics of ground motion, it is not unlikely for the block to experience extended periods of rest during the response.

### *Rock Mode*

This mode is characterized by a rotation about either corner  $O$  or  $O'$  provided that friction is sufficient enough to prevent sliding ( $|f_x| \leq \mu_s f_y$ ). A rock mode (Figure 3.3) is initiated from rest once the overturning moment of the horizontal inertial force about one corner exceeds the restoring moment due to the weight of the block and vertical inertial force:

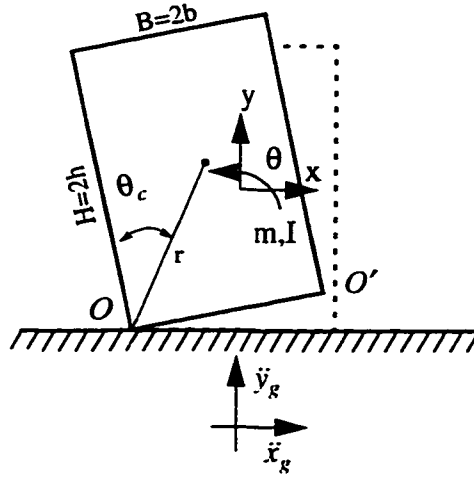
$$m\ddot{x}_g h > mgb + m\ddot{y}_g b$$

or

$$\ddot{x}_g > (\ddot{y}_g + g)b/h \quad (3.4)$$

In the special case that  $\ddot{y}_g = 0$ , Equation (3.4) reduces to

$$|\ddot{x}_g| > gb/h$$



**Fig. 3.3** Rock Mode of a Rigid Block

Once rocking has been initiated, the equations of motion governing the angle  $\theta$  from the vertical are derived by considering the equilibrium of moments about the centers of rotation  $O$  or  $O'$ :

$$I_0 \ddot{\theta} = mr \cos(\theta_c - |\theta|) \ddot{x}_g - mr S(\theta) \sin(\theta_c - |\theta|) (\ddot{y}_g + g), \quad (3.5)$$

where  $I_0$  is the mass moment of inertia of the block about the corner edge  $O$  and  $\theta_c = \text{atan}(B/H)$ .

Clearly, Equation (3.5) is highly nonlinear and not amenable to exact closed-form solutions. However, assuming slender blocks ( $H/B > 3$ ) and small angles of rotation, the above equation can be linearized (e.g., Housner 1963, or Spanos and Koh 1984). Namely, for

$(\theta_c \pm \theta)$  being small, the piecewise linearization procedure yields

$$I_0 \ddot{\theta} = mr\ddot{x}_g - mrS(\theta)(\theta_c - |\theta|)(\ddot{y}_g + g) \quad (3.6)$$

Equation (3.6) is valid as long as  $\theta \neq 0$ . When  $\theta = 0$ , an impact occurs between the block and the foundation. Depending on the assumed impact model, the post-impact motion may be a pure rock about either corner, or possibly another response mode altogether. Previous investigations have focused primarily on rocking behavior. The landmark work by Housner (1963), was based on the two-fold assumption of point contact occurring during a perfectly plastic impact. This assumption guarantees a post-impact rocking response, provided that the static friction coefficient is large enough to prevent any sliding displacement of the contact point, and results in a reduction of angular velocity following the impact, given by

$$\dot{\theta}^+ = e\dot{\theta}^-, \quad (3.7)$$

in which  $e$  is the coefficient of restitution,  $\dot{\theta}^-$  and  $\dot{\theta}^+$  are the angular velocities immediately before and after impact, respectively. The coefficient of restitution is usually determined by considering the conservation of angular momentum or by means of a prescribed reduction in kinetic energy.

If the impact is assumed to be such that there is no bouncing of the block, the rotation continues smoothly and the angular momentum is conserved. Equating the angular momentum about  $O'$  immediately before impact to that immediately after impact gives

$$I_0 \dot{\theta}^- - 2mrbs \sin \theta_c \dot{\theta}^- = I_0 \dot{\theta}^+, \quad (3.8)$$



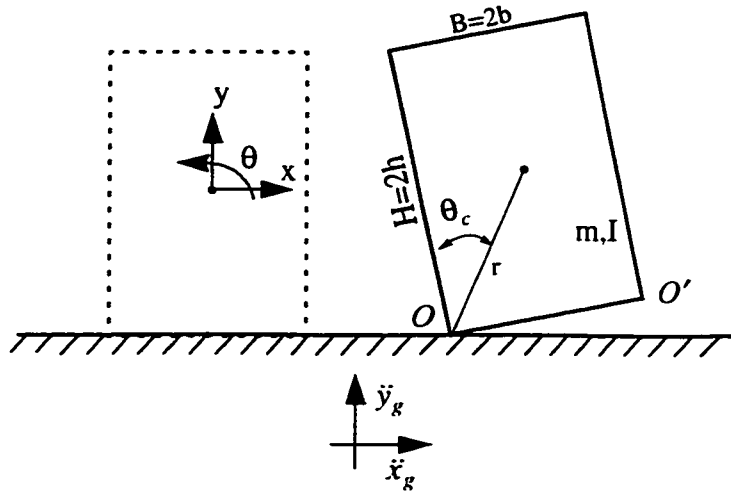
which combined with Equation (3.7), and noting that for a rectangular block  $I_0 = \frac{4}{3}mr^2$ , yields

$$e = 1 - \frac{3}{2} \sin^2 \theta_c. \quad (3.9)$$

It is worth noting that if  $e$  is not computed based on Equation (3.9), the impact cannot be perfectly plastic, in which case, a post-impact rock mode is not guaranteed.

### *Slide-Rock Mode*

In this mode the block rotates about one corner, and simultaneously slides relative to the foundation (Figure 3.4). The friction force is a function of the normal reaction and the velocity of the corner in contact,  $\dot{x}_0$  or  $\dot{x}_0'$ .



**Fig. 3.4** Slide-Rock Mode of a Rigid Block

The conditions governing the initiation of a slide-rock mode from rest are not altogether clear. The mode is perhaps initiated from rest in the singular case:

$$|\ddot{x}_g| > \mu_s(\ddot{y}_g + g) = \tan\theta_c(\ddot{y}_g + g) \quad (3.10)$$

A slide-rock mode, however, may be initiated by transition from another mode of response, e.g., from a rock mode when friction is not sufficient.

The equations of motion of the block may be derived using D'Alempert's principle (Mochizuki and Kobayashi, 1976) or Lagrange's method. The resulting equations are

$$m(\ddot{x} + \ddot{x}_g) = -S(\dot{x}_0)\mu f_y \quad (3.11)$$

$$m(\ddot{y} + \ddot{y}_g + g) = f_y \quad (3.12)$$

$$I_c \ddot{\theta} = -r f_y [S(\theta) \sin(\theta_c - |\theta|) + S(\dot{x}_0)\mu \cos(\theta_c - |\theta|)], \quad (3.13)$$

where  $f_y$  is the normal reaction at the corner in contact with the foundation,  $\mu$  is the dynamic coefficient of friction between the edge in contact and the foundation, and  $S(u)$  is the signum function defined by

$$S(u) = \begin{cases} 1 & u > 0 \\ -1 & u < 0 \end{cases} \quad (3.14)$$

Note that  $y$  is not an independent variable, but rather a function of  $\theta$ , namely,

$$y = r[\cos(\theta_c - |\theta|) - \cos\theta_c]. \quad (3.15)$$

Differentiating Equation (3.15) twice with respect to time yields the vertical acceleration:

$$\dot{y} = S(\theta)r\sin(\theta_c - |\theta|)\dot{\theta}, \quad (3.16)$$

$$\ddot{y} = S(\theta)r\sin(\theta_c - |\theta|)\ddot{\theta} - r\cos(\theta_c - |\theta|)\dot{\theta}^2. \quad (3.17)$$

Substituting Equation (3.17) into Equation (3.12), and eliminating  $f_y$  from Equations (3.11) and (3.13) yields

$$\begin{aligned} & \ddot{x} + S(\theta)S(\dot{x}_0)\mu r \sin(\theta_c - |\theta|)\ddot{\theta} - S(\dot{x}_0)\mu r \cos(\theta_c - |\theta|)\dot{\theta}^2 \\ & = -S(\dot{x}_0)\mu(\ddot{y}_g + g) - \ddot{x}_g \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} & [I_c + mr^2(\sin(\theta_c - |\theta|))^2 + S(\theta)S(\dot{x}_0)\mu mr^2 \sin(\theta_c - |\theta|)\cos(\theta_c - |\theta|)]\ddot{\theta} \\ & - mr^2[S(\theta)\sin(\theta_c - |\theta|)\cos(\theta_c - |\theta|) + S(\dot{x}_0)\mu(\cos(\theta_c - |\theta|))^2]\dot{\theta}^2 \\ & + mr[S(\theta)\sin(\theta_c - |\theta|) + S(\dot{x}_0)\mu\cos(\theta_c - |\theta|)](\ddot{y}_g + g) = 0 \end{aligned} \quad (3.19)$$

In order to evaluate  $S(\dot{x}_0)$ , the velocity of the corner in contact must be determined.

Expressing the relative displacement of the corner  $x_0$ , in terms of  $x$  and  $\theta$  gives

$$x_0 = x - S(\theta)r\sin(\theta_c - |\theta|). \quad (3.20)$$

Differentiating Equation (3.20) with respect to time leads to

$$\dot{x}_0 = \dot{x} + r\cos(\theta_c - |\theta|)\dot{\theta} \quad (3.21)$$

and

$$\ddot{x}_0 = \ddot{x} + r\cos(\theta_c - |\theta|)\ddot{\theta} + S(\theta)r\sin(\theta_c - |\theta|)\dot{\theta}^2. \quad (3.22)$$

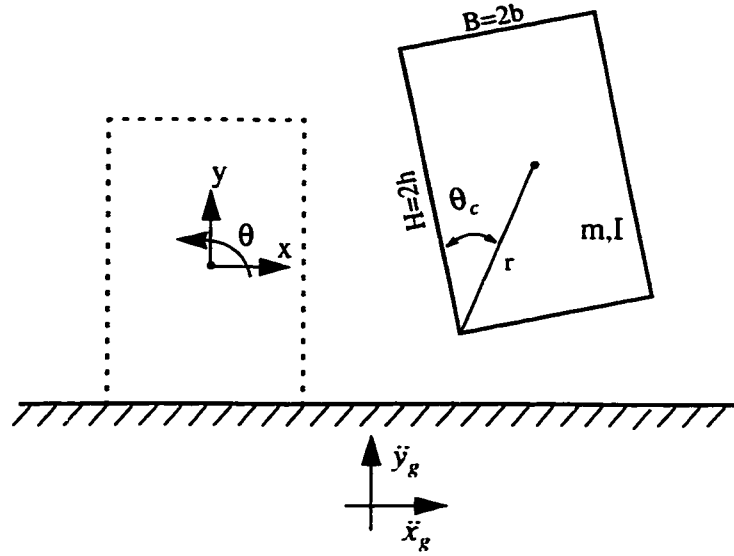
Equations (3.20) through (3.22) give the displacement, velocity and acceleration of corner  $O$  for  $\theta > 0$  ( $S(\theta) = +1$ ) and of corner  $O'$  for  $\theta < 0$  ( $S(\theta) = -1$ ).

As with the rock mode, Equations (3.18) and (3.19) are valid, provided  $\theta \neq 0$ . When  $\theta = 0$ , an impact occurs between the block and the foundation. The analysis of impact in this case is much more involved than that of the rocking case.

### *Free-Flight Mode*

Free-flight occurs when the body loses all contact with the foundation. That is, when the

normal reaction force equals zero (Figure 3.5).



**Fig. 3.5** Free-Flight Mode of a Rigid Block

Assuming the body is initially at rest or in a slide mode, free-flight is initiated when

$$\ddot{y}_g(t) = -g. \quad (3.23)$$

For a body in a rock or slide-rock mode, the normal force is found by substituting Equation (3.17) into Equation (3.12). That is,

$$f_y = m[rS(\theta)\sin(\theta_c - |\theta|)\ddot{\theta} - r\cos(\theta_c - |\theta|)\dot{\theta}^2 + \ddot{y}_g + g]. \quad (3.24)$$

Initiation of free-flight is realized when  $f_y = 0$ , namely,

$$\ddot{y}_g = -rS(\theta)\sin(\theta_c - |\theta|)\ddot{\theta} + r\cos(\theta_c - |\theta|)\dot{\theta}^2 - g. \quad (3.25)$$

Once free-flight has been initiated, the equations of motion are

$$m(\ddot{x} + \ddot{x}_g) = 0 \quad (3.26)$$

$$m(\ddot{y} + \ddot{y}_g + g) = 0 \quad (3.27)$$

$$I\ddot{\theta} = 0 \quad (3.28)$$

Note that these equations may be integrated readily for known accelerations and initial conditions. Equations (3.26) through (3.28) are valid until a point of the body comes in contact with the foundation. At this time, an impact occurs which can be handled using the principle of impulse and momentum, with the effects of restitution and friction included.

### 3.3 Analysis of Rocking Systems

The equations of motion governing the rock mode are highly nonlinear and, therefore, closed-form solutions are extremely difficult to obtain, even for the simplest of ground motions. For this reason, assumptions are commonly made in order to simplify the governing equation, restrict the motion to a single mode, or both. Typical assumptions made with regard to rocking include small angles of rotation, slender blocks (e.g.,  $H/B > 3$ ), sufficient friction to prevent sliding, and perfectly plastic impacts. The latter two assumptions prohibit a transition to a slide, slide-rock, or free-flight mode. In many cases, however, the validity of these assumptions has not been investigated.

#### 3.3.1 Equation of Motion

The equation of motion governing the response of a symmetric rigid block is given by

Equation (3.5). For a rectangular block, for which  $I_0 = \frac{4}{3}mr^2$ , this reduces to

$$\ddot{\theta} = \frac{3}{4r} \cos(\theta_c - |\theta|) \ddot{x}_g - \frac{3}{4r} S(\theta) \sin(\theta_c - |\theta|) (\ddot{y}_g + g). \quad (3.29)$$

Expanding the trigonometric terms in Equation (3.29) and neglecting  $\ddot{y}_g$  leads to

$$\ddot{\theta} = \frac{3}{4r}(\cos\theta_c \cos|\theta| + \sin\theta_c \sin|\theta|)\ddot{x}_g - \frac{3g}{4r}S(\theta)(\sin\theta_c \cos|\theta| - \cos\theta_c \sin|\theta|) \quad (3.30)$$

Assuming small angles of rotation, such that  $\sin\theta \approx \theta$  and  $\cos\theta \approx 1$ , Equation (3.30) reduces to

$$\ddot{\theta} = \frac{3}{4r}(\cos\theta_c + \sin\theta_c|\theta|)\ddot{x}_g - \frac{3g}{4r}S(\theta)(\sin\theta_c - \cos\theta_c|\theta|). \quad (3.31)$$

A further approximation can be made assuming  $|\theta| \ll H/B$ , which implies  $\sin\theta_c|\theta| \ll \cos\theta_c$ . That is

$$\ddot{\theta} = \frac{3}{4r}\cos\theta_c\ddot{x}_g - \frac{3g}{4r}S(\theta)(\sin\theta_c - \cos\theta_c|\theta|),$$

or

$$\ddot{\theta} - \frac{3g}{4r}\cos\theta_c\theta = \frac{3}{4r}\cos\theta_c\ddot{x}_g - \frac{3g}{4r}S(\theta)\sin\theta_c. \quad (3.32)$$

It should be noted that Equation (3.32) reduces to Equation (3.6), in which  $(\theta_c - |\theta|)$  was assumed small, if  $\theta_c$  is also assumed small. In other words, Equation (3.32) provides further accuracy for aspect ratios  $H/B < 3$ .

For consistency, dimensionless equations are sought and the following quantities are introduced

$$\alpha^2 = \frac{mgr}{I_0}, \quad (3.33)$$

$$\Theta = \frac{\theta}{\theta_c}, \quad (3.34)$$

and

$$\tau = \alpha t. \quad (3.35)$$

Applying Equations (3.33) through (3.35) into Equation (3.32) yields

$$\ddot{\Theta} - \cos\theta_c \Theta = \frac{1}{\theta_c} [-S(\Theta) \sin\theta_c + \cos\theta_c \ddot{x}_g / g]. \quad (3.36)$$

Furthermore, introducing

$$\beta^2 = \cos\theta_c, \quad (3.37)$$

and

$$\gamma = \tan\theta_c / \theta_c, \quad (3.38)$$

Equation (3.36) is transformed to

$$\ddot{\Theta} - \beta^2 \Theta = -\beta^2 \gamma S(\Theta) + \frac{\beta^2}{g\theta_c} \ddot{x}_g. \quad (3.39)$$

Equation (3.39) is a nonhomogeneous nonlinear ordinary differential equation with constant coefficients. It is also referred to as “piecewise linear”, since for  $S(\Theta) = \pm 1$ , the governing equation is linear.

### 3.3.2 Free Rocking

When the block is rotated through an angle  $\theta_0$  and released with initial displacement, it will be set to rocking about  $O$  and  $O'$  until the motion decays to rest. The equation governing the free vibration is obtained from Equation (3.40) with the base acceleration set to zero. That is,

$$\ddot{\Theta} - \beta^2 \Theta = -\gamma \beta^2 S(\Theta). \quad (3.40)$$

The general solution of Equation (3.39), for  $\Theta > 0$ , is given by

$$\Theta(\tau) = C_1 \cosh \beta \tau + C_2 \sinh \beta \tau + \gamma. \quad (3.41)$$

Equation (3.41) subject to the initial conditions  $\Theta(0) = \Theta_0$  and  $\dot{\Theta}(0) = 0$  yields

$$\Theta(\tau) = (\Theta_0 - \gamma) \cosh \beta \tau + \gamma, \quad (3.42)$$

which is valid for  $0 \leq \tau \leq \tau_1$ , in which  $\tau_1$  denotes the time to first impact.

The time  $\tau_1$  to the first impact is determined from Equation (3.42) by setting  $\Theta(\tau) = 0$ :

$$\tau_1 = \frac{1}{\beta} \operatorname{acosh} \left( \frac{\gamma}{\gamma - \Theta_0} \right) \quad (3.43)$$

After the first impact, the block will rotate about  $O'$ , and if there is no energy loss during impact, it will rise until the angle becomes again equal to  $\theta_0$ . At this instant, half cycle of vibration has been completed. The time  $T$  required to complete this cycle is the period of free vibration, given by

$$T = 4\tau_1 = \frac{4}{\beta} \operatorname{acosh} \left( \frac{\gamma}{\gamma - \Theta_0} \right) \quad (3.44)$$

in which  $\beta$  and  $\gamma$  are functions of  $\theta_c$ , as defined in Equations (3.37) and (3.38).

Assuming sufficient friction to prevent sliding, impacts are governed by Equations (3.7) and (3.9). Specifically, following the first impact the governing equation of motion is described by Equation (3.39) taking into account the change in sign of  $\Theta$  and the new initial conditions given by

$$\Theta(\tau_1) = 0 \quad \text{and} \quad \dot{\Theta}(\tau_1^+) = e\dot{\Theta}(\tau_1^-). \quad (3.45)$$

The time to next impact,  $\tau_2$ , is then determined and the procedure continues.



## **CHAPTER 4: Dynamic Analysis of Stacked Rigid Blocks**

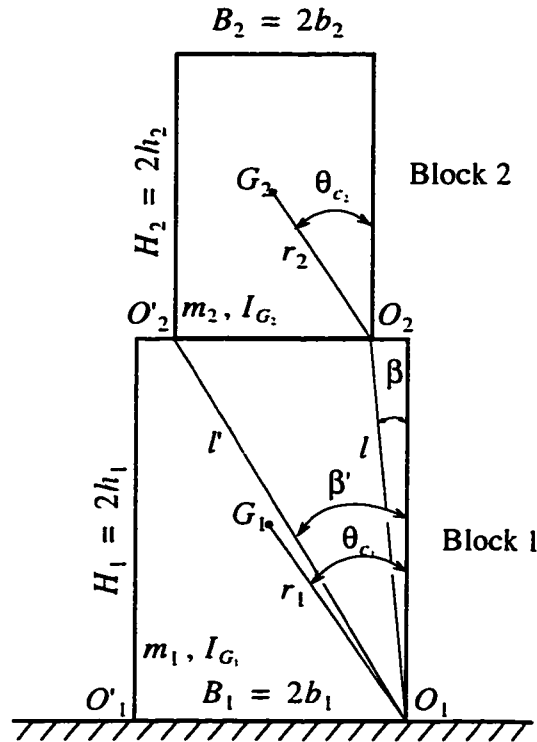
### **4.1 Introduction**

As opposed to the case of a single rigid block, the dynamic behavior of multi-block structures has not, to date, been adequately studied. Even for the simplest case of multi-block structures, i.e., two-block assemblies, the analytical formulation of the problem is quite challenging. Its complexity is associated with the continuous transition from one regime of motion to another, each one being governed by a different set of highly nonlinear equations.

In this chapter, the dynamic behavior of systems consisting of two blocks, one placed over the other, is examined. Such configuration can be thought of as the model of ancient Greek and Roman type structures composed of large heavy members that simply lie on top of each other, or a piece of machinery or statue placed on top of a block-like base. Assuming no sliding, the rocking response of the system standing free on a rigid foundation is investigated. In particular, all different modes of response are classified and criteria for initiation of motion are derived. For each mode of response, the full nonlinear equations of motion are formulated and transition criteria between modes, in the absence of impact, are outlined. The impact problem is addressed in Chapter 5.

### **4.2 Analytical Model**

The system under consideration is presented schematically in Figure 4.1. It consists of two symmetric rigid blocks; the upper block resting symmetrically on top of the lower block, and the latter resting on a rigid horizontal surface. Their mass is denoted by  $m_i$  and their centroid moment of inertia by  $I_{G_i}$ , where  $i$  is the block index.

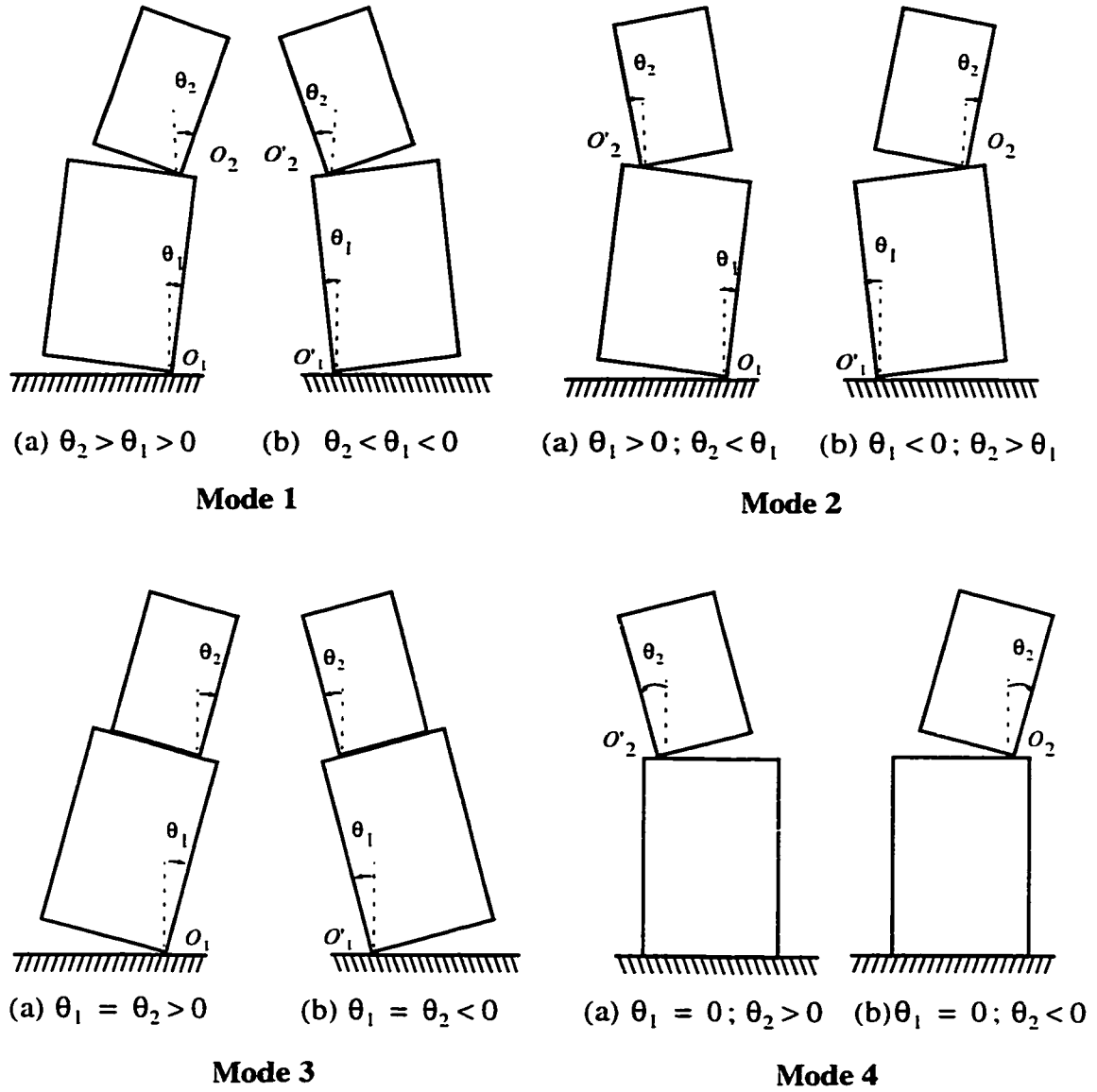


**Fig. 4.1** Geometric Model of Two Stacked Rigid Blocks

Assuming that sufficient friction exists to prevent sliding of either block, the system possesses two degrees of freedom, namely,  $\theta_1$  and  $\theta_2$  denoting the angles of rotation of Block 1 and Block 2 from the vertical.

When subjected to a base acceleration with horizontal and vertical components  $\ddot{x}_g$  and  $\ddot{y}_g$  respectively, the block may exhibit four possible regimes of rocking motion, henceforth called 'modes' for simplicity (Figure 4.2). The first two modes reflect a two-degree-of-freedom system response, and they are discretized depending on whether the blocks rotate in the same or opposite direction. Modes 3 and 4 reflect a single-degree-of-freedom system response. In particular, mode 3 describes the motion of the system rocking as one rigid structure ( $\theta_1 = \theta_2$ ), and mode 4 concerns the case where only one of the blocks expe-

riences rotation ( $\theta_1 = 0$ ,  $\theta_2 \neq 0$ ). Furthermore, each mode is divided into two subcases to account for the change in angle sign.

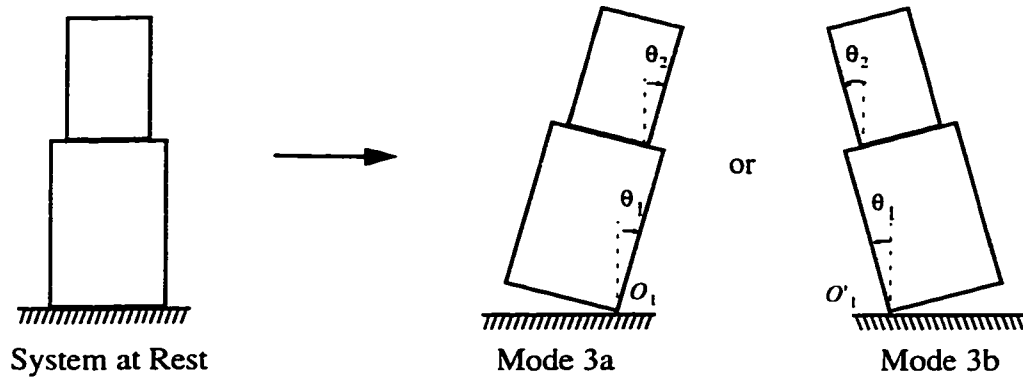


**Fig. 4.2** Classification of Rocking Modes for a System of Two Stacked Rigid Blocks

### 4.3 Analysis of Rocking

#### 4.3.1 Initiation of Motion

When subjected to a base acceleration with horizontal and vertical components  $\ddot{x}_g$  and  $\ddot{y}_g$  respectively, the system may be set into rocking in either mode 3, or Mode 4, (Figures 4.3 and 4.4), when the overturning moment of the horizontal inertia force about one edge exceeds the restoring moment due to the weight(s) of the block(s) and the vertical inertia force.



**Fig. 4.3** Initiation of Motion in Mode 3

The criteria for initiation of motion are given below.

Transition from rest to mode 3a requires

$$-h\ddot{x}_g - b_1\ddot{y}_g > b_1g, \quad (4.1)$$

where  $h = \frac{m_1h_1 + m_2(2h_1 + h_2)}{m_1 + m_2}$  is the distance of the mass center of the system from the

base of block 1.

Transition from rest to mode 3b requires

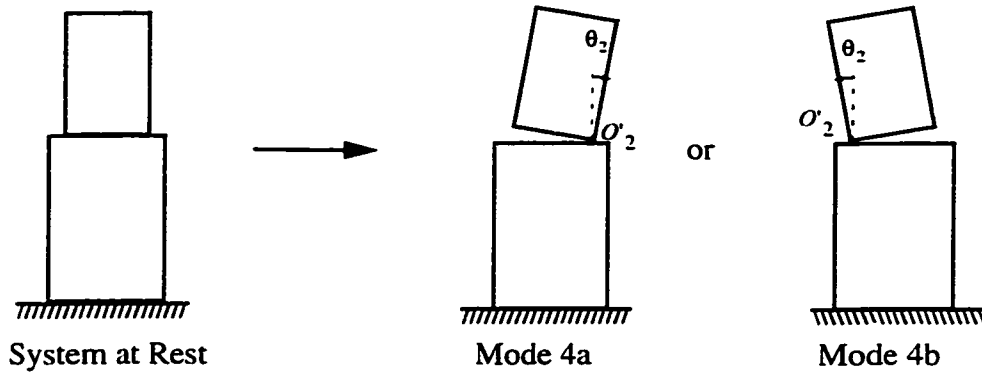
$$h\ddot{x}_g - b_1\ddot{y}_g > b_1g. \quad (4.2)$$

Transition from rest to mode 4a requires

$$-h_2\ddot{x}_g - b_2\ddot{y}_g > b_2g. \quad (4.3)$$

Finally, transition from rest to mode 4b requires

$$h_2\ddot{x}_g - b_2\ddot{y}_g > b_2g. \quad (4.4)$$



**Fig. 4.4** Initiation of Motion in Mode 4

#### 4.3.2 Formulation of the Equations of Motion

The equations of motion of the system may be derived applying Newton's second law to each block separately, or Lagrange's method to the system as a whole. Herein, for the two-degree-of-freedom-system response, Lagrange's equations are used, which eliminates the necessity of evaluating the interaction forces, and permits the use of scalar quantities, work and kinetic energy, instead of the vector quantities, force and displacements.

The well-known Lagrange's equations of motion for an N-degree-of-freedom system are expressed in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i; \quad i = 1, 2, \dots, N \quad (4.5)$$

in which

$q_1, q_2, \dots, q_N$  is a set of generalized coordinates,  $T = T(q_1, q_2, \dots, q_N, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_N)$  is the total kinetic energy expressed in terms of the generalized coordinates and their first time derivatives,  $V = V(q_1, q_2, \dots, q_N)$  the potential energy expressed in terms of the generalized coordinates, and  $Q_1, Q_2, \dots, Q_N$  are the non-conservative generalized forcing functions corresponding to the coordinates  $q_1, q_2, \dots, q_N$ , respectively; they are defined by the equation

$$\delta W_{nc} = Q_1 \delta q_1 + Q_2 \delta q_2 + \dots + Q_N \delta q_N, \quad (4.6)$$

in which  $\delta W_{nc}$  is the virtual work performed by the nonconservative forces as they act through the virtual displacements caused by an arbitrary set of variations in the generalized coordinates.

For the system under consideration, the angles of rotation of block 1 and block 2 from the vertical are chosen for the generalized coordinates of the system, so that the governing equations of motion may be expressed in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right) - \frac{\partial T}{\partial \theta_i} + \frac{\partial V}{\partial \theta_i} = 0; \quad i = 1, 2 \quad (4.7)$$

where

$$T = T_1 + T_2 = \frac{1}{2} \sum_{i=1}^2 (m_i v_{G_i}^2 + I_{G_i} \dot{\theta}_i^2) \quad (4.8)$$

is the kinetic energy of the system, and,

$$V = V_1 + V_2 = \sum_{i=1}^2 (m_i h_{G_i} g) \quad (4.9)$$

is the gravitational potential energy of the system. In the above equations,  $I_{G_i}$  denotes the centroid moment of inertia of the  $i^{th}$  block,  $v_{G_i}$  the velocity of the center of mass,  $\dot{\theta}_i$  the angular velocity of the block, and  $h_{G_i}$  the location of the center of mass above the ground. Note that, since the system is conservative, the generalized forces do not appear in Equation (4.7):  $Q_i = 0$ .

#### *Governing Equations for Mode I*

In the first mode, the blocks are rotating in the same direction. However, it is necessary to differentiate among the two subcases of opposite angle sign (Figure 4.2), because different equations of motion hold in each subcase.

Consider first the case where the system is rocking in mode 1a. Assuming that block 1 always remains in contact with the foundation, the global coordinates of its mass center  $G_1$ , are related to the excitation displacement components by the equations

$$\begin{aligned} x_{G_1} &= x_g - r_1 \sin(\theta_{c_1} - \theta_1) \\ y_{G_1} &= y_g + r_1 \cos(\theta_{c_1} - \theta_1). \end{aligned} \quad (4.10)$$

Differentiating the above equations with respect to time yields

$$\begin{aligned} \dot{x}_{G_1} &= \dot{x}_g + r_1 \dot{\theta}_1 \cos(\theta_{c_1} - \theta_1) \\ \dot{y}_{G_1} &= \dot{y}_g + r_1 \dot{\theta}_1 \sin(\theta_{c_1} - \theta_1). \end{aligned} \quad (4.11)$$

The velocity of the mass center of block 1 is given by

$$\begin{aligned}
v_{G_1}^2 &= \dot{x}_{G_1}^2 + \dot{y}_{G_1}^2 \\
&= r_1^2 \dot{\theta}_1^2 + \dot{x}_g^2 + \dot{y}_g^2 + 2r_1 \dot{\theta}_1 [\dot{x}_g \cos(\theta_{c_1} - \theta_1) + \dot{y}_g \sin(\theta_{c_1} - \theta_1)].
\end{aligned} \tag{4.12}$$

Similarly, the global coordinates of the mass center  $G_2$  of block 2, are

$$\begin{aligned}
x_{G_2} &= x_{O_2} - r_2 \sin(\theta_{c_2} - \theta_2) \\
y_{G_2} &= y_{O_2} + r_2 \cos(\theta_{c_2} - \theta_2),
\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}
x_{O_2} &= x_g + l \sin(\theta_1 - \beta) \\
y_{O_2} &= y_g + l \cos(\theta_1 - \beta).
\end{aligned} \tag{4.14}$$

Differentiating the above equations with respect to time yields

$$\begin{aligned}
\dot{x}_{G_2} &= \dot{x}_g + l \dot{\theta}_1 \cos(\theta_1 - \beta) + r_2 \dot{\theta}_2 \cos(\theta_{c_2} - \theta_2) \\
\dot{y}_{G_2} &= \dot{y}_g - l \dot{\theta}_1 \sin(\theta_1 - \beta) + r_2 \dot{\theta}_2 \sin(\theta_{c_2} - \theta_2).
\end{aligned} \tag{4.15}$$

The velocity of the mass center of block 2 is then derived by

$$\begin{aligned}
v_{G_2}^2 &= \dot{x}_{G_2}^2 + \dot{y}_{G_2}^2 \\
&= \dot{x}_g^2 + \dot{y}_g^2 + l^2 \dot{\theta}_1^2 + r_2^2 \dot{\theta}_2^2 + 2l \dot{\theta}_1 [\dot{x}_g \cos(\theta_1 - \beta) - \dot{y}_g \sin(\theta_1 - \beta)] + \\
&\quad + 2r_2 \dot{\theta}_2 [\dot{x}_g \cos(\theta_{c_2} - \theta_2) + \dot{y}_g \sin(\theta_{c_2} - \theta_2)] + 2r_2 l \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2 + \theta_{c_2} - \beta).
\end{aligned} \tag{4.16}$$

Substituting Equations (4.12) and (4.16) into Equation (4.8), leads to the following expression for the kinetic energy of the system

$$\begin{aligned}
T &= \frac{1}{2} m_1 \{ r_1^2 \dot{\theta}_1^2 + \dot{x}_g^2 + \dot{y}_g^2 + 2r_1 \dot{\theta}_1 [\dot{x}_g \cos(\theta_{c_1} - \theta_1) + \dot{y}_g \sin(\theta_{c_1} - \theta_1)] \} + \\
&\quad + \frac{1}{2} I_{G_1} \dot{\theta}_1^2 + \frac{1}{2} m_2 \{ \dot{x}_g^2 + \dot{y}_g^2 + l^2 \dot{\theta}_1^2 + r_2^2 \dot{\theta}_2^2 + 2l \dot{\theta}_1 [\dot{x}_g \cos(\theta_1 - \beta) - \\
&\quad - \dot{y}_g \sin(\theta_1 - \beta)] + 2r_2 \dot{\theta}_2 [\dot{x}_g \cos(\theta_{c_2} - \theta_2) + \dot{y}_g \sin(\theta_{c_2} - \theta_2)] + \\
&\quad + 2r_2 l \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2 + \theta_{c_2} - \beta) \} + \frac{1}{2} I_{G_2} \dot{\theta}_2^2.
\end{aligned} \tag{4.17}$$



The gravitational potential energy of the system rocking in mode 1a is obtained from Equation (4.9). Specifically,

$$V = m_1 g r_1 \cos(\theta_{c_1} - \theta_1) + m_2 g [r_2 \cos(\theta_{c_2} - \theta_2) + l \cos(\theta_1 - \beta)]. \quad (4.18)$$

Equations (4.17) and (4.18) are used to obtain the derivatives necessary for use in Lagrange's Equation (4.7). That is,

$$\begin{aligned} \frac{\partial T}{\partial \dot{\theta}_1} = & (I_{O_1} + m_2 l^2) \dot{\theta}_1 + m_2 r_2 l \cos(\theta_1 - \theta_2 + \theta_{c_2} - \beta) \dot{\theta}_2 + m_1 r_1 [\dot{x}_g \cos(\theta_{c_1} - \theta_1) + \\ & + \dot{y}_g \sin(\theta_{c_1} - \theta_1)] + m_2 l [\dot{x}_g \cos(\theta_1 - \beta) - \dot{y}_g \sin(\theta_1 - \beta)], \end{aligned}$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = m_2 r_2 l \cos(\theta_1 - \theta_2 + \theta_{c_2} - \beta) \dot{\theta}_1 + I_{O_2} \dot{\theta}_2 + m_2 r_2 [\dot{x}_g \cos(\theta_{c_2} - \theta_2) + \dot{y}_g \sin(\theta_{c_2} - \theta_2)],$$

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_1} \right) = & (I_{O_1} + m_2 l^2) \ddot{\theta}_1 + m_2 r_2 l \cos \gamma_{1a} \ddot{\theta}_2 - m_2 r_2 l \sin \gamma_{1a} (\dot{\theta}_1 - \dot{\theta}_2) \dot{\theta}_2 + m_1 r_1 + \\ & [\ddot{x}_g \cos(\theta_{c_1} - \theta_1) + \dot{x}_g \sin(\theta_{c_1} - \theta_1) \dot{\theta}_1 + \ddot{y}_g \sin(\theta_{c_1} - \theta_1) - \dot{y}_g \cos(\theta_{c_1} - \theta_1) \dot{\theta}_1] \\ & + m_2 l [\ddot{x}_g \cos(\theta_1 - \beta) - \dot{x}_g \sin(\theta_1 - \beta) \dot{\theta}_1 - \ddot{y}_g \sin(\theta_1 - \beta) - \dot{y}_g \cos(\theta_1 - \beta) \dot{\theta}_1], \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_2} \right) = & m_2 r_2 l \cos \gamma_{1a} \ddot{\theta}_1 - m_2 r_2 l \dot{\theta}_1 \sin \gamma_{1a} (\dot{\theta}_1 - \dot{\theta}_2) + I_{O_2} \ddot{\theta}_2 + m_2 r_2 [\ddot{x}_g \cos(\theta_{c_2} - \theta_2) + \\ & + \dot{x}_g \sin(\theta_{c_2} - \theta_2) \dot{\theta}_2 + \ddot{y}_g \sin(\theta_{c_2} - \theta_2) - \dot{y}_g \cos(\theta_{c_2} - \theta_2) \dot{\theta}_2], \end{aligned}$$

$$\begin{aligned} \frac{\partial T}{\partial \theta_1} = & m_1 r_1 \dot{\theta}_1 [\dot{x}_g \sin(\theta_{c_1} - \theta_1) - \dot{y}_g \cos(\theta_{c_1} - \theta_1)] - m_2 l \dot{\theta}_1 [\dot{x}_g \sin(\theta_1 - \beta) + \dot{y}_g \cos(\theta_1 - \beta)] \\ & - m_2 r_2 l \dot{\theta}_1 \dot{\theta}_2 \sin \gamma_{1a}, \end{aligned}$$

$$\frac{\partial T}{\partial \theta_2} = m_2 r_2 \dot{\theta}_2 [\dot{x}_g \sin(\theta_{c_2} - \theta_2) - \dot{y}_g \cos(\theta_{c_2} - \theta_2)] + m_2 r_2 l \dot{\theta}_1 \dot{\theta}_2 \sin \gamma_{1a},$$

$$\frac{\partial V}{\partial \theta_1} = m_1 g r_1 \sin(\theta_{c_1} - \theta_1) - m_2 g l \sin(\theta_1 - \beta), \text{ and}$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g r_2 \sin(\theta_{c_1} - \theta_2).$$

The governing equations for rocking response in mode 1a are then constructed using Equation (4.7). This procedure gives,

$$\left. \begin{aligned} (I_{O_1} + m_2 l^2) \ddot{\theta}_1 + m_2 r_2 l \cos \gamma_{1a} \ddot{\theta}_2 + m_2 r_2 l \sin \gamma_{1a} \dot{\theta}_2^2 - m_1 r_1 g \sin(\theta_1 - \theta_{c_1}) \\ - m_2 g l \sin(\theta_1 - \beta) = -[m_1 r_1 \cos(\theta_1 - \theta_{c_1}) + m_2 l \cos(\theta_1 - \beta)] \ddot{x}_g + \\ + [m_1 r_1 \sin(\theta_1 - \theta_{c_1}) + m_2 l \sin(\theta_1 - \beta)] \ddot{y}_g \\ m_2 r_2 l \cos \gamma_{1a} \ddot{\theta}_1 + I_{O_2} \ddot{\theta}_2 - m_2 r_2 l \sin \gamma_{1a} \dot{\theta}_1^2 - m_2 g r_2 \sin(\theta_2 - \theta_{c_1}) \\ = m_2 r_2 [\ddot{y}_g \sin(\theta_2 - \theta_{c_1}) - \ddot{x}_g \cos(\theta_2 - \theta_{c_1})] \end{aligned} \right\} \quad (4.19)$$

which is valid for  $\theta_2 > \theta_1 > 0$ . In the above equation  $\gamma_{1a}$  is defined by

$$\gamma_{1a} = \theta_1 - \theta_2 + \theta_{c_1} - \beta. \quad (4.20)$$

For rocking response in mode 1b, the governing equations are similarly derived and can be written in the form

$$\left. \begin{aligned} (I_{O_1} + m_2 l^2) \ddot{\theta}_1 + m_2 r_2 l \cos \gamma_{1b} \ddot{\theta}_2 + m_2 r_2 l \sin \gamma_{1b} \dot{\theta}_2^2 - m_1 g r_1 \sin(\theta_1 + \theta_{c_1}) \\ - m_2 g l \sin(\theta_1 + \beta) = -[m_1 r_1 \cos(\theta_1 + \theta_{c_1}) + m_2 l \cos(\theta_1 + \beta)] \ddot{x}_g + \\ + [m_1 r_1 \sin(\theta_1 + \theta_{c_1}) + m_2 l \sin(\theta_1 + \beta)] \ddot{y}_g \\ m_2 r_2 l \cos \gamma_{1b} \ddot{\theta}_1 + I_{O_2} \ddot{\theta}_2 - m_2 r_2 l \sin \gamma_{1b} \dot{\theta}_1^2 - m_2 g r_2 \sin(\theta_2 + \theta_{c_1}) \\ = m_2 r_2 [\ddot{y}_g \sin(\theta_2 + \theta_{c_1}) - \ddot{x}_g \cos(\theta_2 + \theta_{c_1})] \end{aligned} \right\} \quad (4.21)$$

which is valid for  $\theta_2 < \theta_1 < 0$ , and  $\gamma_{1b}$  is defined by the equation

$$\gamma_{1b} = \theta_1 - \theta_2 - \theta_{c_2} + \beta. \quad (4.22)$$

Combining Equations (4.19) through (4.22), leads to a compact set of equations for mode 1, namely,

$$\left. \begin{aligned} (I_{O_1} + m_2 l^2) \ddot{\theta}_1 + m_2 r_2 l \cos \gamma_1 \ddot{\theta}_2 + m_2 r_2 l \sin \gamma_1 \dot{\theta}_2^2 - m_1 r_1 g \sin(\theta_1 - S_{\theta_1} \theta_{c_1}) \\ - m_2 l g \sin(\theta_1 - S_{\theta_1} \beta) = -[m_1 r_1 \cos(\theta_1 - S_{\theta_1} \theta_{c_1}) + m_2 l \cos(\theta_1 - S_{\theta_1} \beta)] \ddot{x}_g \\ + [m_1 r_1 \sin(\theta_1 - S_{\theta_1} \theta_{c_1}) + m_2 l \sin(\theta_1 - S_{\theta_1} \beta)] \ddot{y}_g \\ m_2 r_2 l \cos \gamma_1 \ddot{\theta}_1 + I_{O_2} \ddot{\theta}_2 - m_2 r_2 l \sin \gamma_1 \dot{\theta}_1^2 - m_2 g r_2 \sin(\theta_2 - S_{\theta_2} \theta_{c_2}) \\ = m_2 r_2 [\ddot{y}_g \sin(\theta_2 - S_{\theta_2} \theta_{c_2}) - \ddot{x}_g \cos(\theta_2 - S_{\theta_2} \theta_{c_2})] \end{aligned} \right\} \quad (4.23)$$

where  $S_{\theta_i}$  denotes the signum function in  $\theta_i$ , and

$$\gamma_1 = \theta_1 - \theta_2 + S_{\theta_1}(\theta_{c_1} - \beta). \quad (4.24)$$

Clearly, Equation (4.23) is highly nonlinear and not amenable to exact closed-form solutions. An approximate solution can, however, be obtained using numerical methods.

### *Governing Equations for Mode 2*

In the second mode, the two blocks are rotating in the opposite direction. Specifically, mode 2a describes the response for  $\theta_1 > 0$  and mode 2b for  $\theta_1 < 0$ .

For mode 2a, the velocity of the mass center of block 1 is the same as that of mode 1a given by Equation (4.6). In order to find the velocity of the mass center of block 2, the coordinates of  $G_2$  are expressed in terms of the coordinates of point  $O_2'$ , namely,

$$\begin{aligned} x_{G_2} &= x_{O_2'} + r_2 \sin(\theta_{c_2} + \theta_2) \\ y_{G_2} &= y_{O_2'} + r_2 \cos(\theta_{c_2} + \theta_2), \end{aligned} \quad (4.25)$$

where

$$\begin{aligned}x_{O_2} &= x_g - l \sin(\beta' - \theta_1) \\y_{O_2} &= y_g + l \cos(\beta' - \theta_1).\end{aligned}\quad (4.26)$$

Differentiating (4.25) with respect to time yields

$$\begin{aligned}\dot{x}_{G_2} &= \dot{x}_g + l' \dot{\theta}_1 \cos(\beta' - \theta_1) + r_2 \dot{\theta}_2 \cos(\theta_{c_2} + \theta_2) \\ \dot{y}_{G_2} &= \dot{y}_g + l' \dot{\theta}_1 \sin(\beta' - \theta_1) - r_2 \dot{\theta}_2 \sin(\theta_{c_2} + \theta_2).\end{aligned}\quad (4.27)$$

The velocity of the mass center of Block 2 is then derived by

$$\begin{aligned}v_{G_2}^2 &= \dot{x}_{G_2}^2 + \dot{y}_{G_2}^2 \\ &= \dot{x}_g^2 + \dot{y}_g^2 + l'^2 \dot{\theta}_1^2 + r_2^2 \dot{\theta}_2^2 + 2l' \dot{\theta}_1 [\dot{x}_g \cos(\beta' - \theta_1) + \dot{y}_g \sin(\beta' - \theta_1)] + \\ &\quad + 2r_2 \dot{\theta}_2 [\dot{x}_g \cos(\theta_{c_2} + \theta_2) - \dot{y}_g \sin(\theta_{c_2} + \theta_2)] + 2r_2 l' \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1 + \theta_{c_2} + \beta').\end{aligned}\quad (4.28)$$

Substituting Equations (4.12) and (4.28) into Equation (4.8), the kinetic energy can be written as

$$\begin{aligned}T &= \frac{1}{2} m_1 \{ r_1^2 \dot{\theta}_1^2 + \dot{x}_g^2 + \dot{y}_g^2 + 2r_1 \dot{\theta}_1 [\dot{x}_g \cos(\theta_{c_1} - \theta_1) + \dot{y}_g \sin(\theta_{c_1} - \theta_1)] \} \\ &\quad + \frac{1}{2} I_{G_1} \dot{\theta}_1^2 + \frac{1}{2} m_2 \{ \dot{x}_g^2 + \dot{y}_g^2 + l'^2 \dot{\theta}_1^2 + r_2^2 \dot{\theta}_2^2 + 2l' \dot{\theta}_1 [\dot{x}_g \cos(\beta' - \theta_1) \\ &\quad - \dot{y}_g \sin(\beta' - \theta_1)] + 2r_2 \dot{\theta}_2 [\dot{x}_g \cos(\theta_{c_2} + \theta_2) - \dot{y}_g \sin(\theta_{c_2} + \theta_2)] \\ &\quad + 2r_2 l' \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1 + \theta_{c_2} + \beta') \} + \frac{1}{2} I_{G_2} \dot{\theta}_2^2.\end{aligned}\quad (4.29)$$

The gravitational potential energy of the system rocking in mode 2a, is obtained from Equation (4.9); the result is

$$V = m_1 g r_1 \cos(\theta_{c_1} - \theta_1) + m_2 g [r_2 \cos(\theta_{c_2} + \theta_2) + l' \cos(\beta' - \theta_1)]. \quad (4.30)$$

Equations (4.29) and (4.30) are used to obtain the derivatives necessary for use in Lagrange's Equation (4.7). Specifically,

$$\begin{aligned} \frac{\partial T}{\partial \dot{\theta}_1} = & (I_{O_1} + m_2 l^2) \dot{\theta}_1 + m_2 r_2 l \cos(\theta_2 - \theta_1 + \theta_{c_1} + \beta') \dot{\theta}_2 + m_1 r_1 [\dot{x}_g \cos(\theta_{c_1} - \theta_1) \\ & + \dot{y}_g \sin(\theta_{c_1} - \theta_1)] + m_2 l' [\dot{x}_g \cos(\beta' - \theta_1) + \dot{y}_g \sin(\beta' - \theta_1)], \end{aligned}$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = m_2 r_2 l \cos(\theta_2 - \theta_1 + \theta_{c_1} + \beta') \dot{\theta}_1 + I_{O_2} \dot{\theta}_2 + m_2 r_2 [\dot{x}_g \cos(\theta_{c_1} + \theta_2) - \dot{y}_g \sin(\theta_{c_1} + \theta_2)],$$

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_1} \right) = & (I_{O_1} + m_2 l^2) \ddot{\theta}_1 + m_2 r_2 l \cos \gamma_{2a} \ddot{\theta}_2 - m_2 r_2 l \sin \gamma_{2a} (\dot{\theta}_2 - \dot{\theta}_1) \dot{\theta}_2 + m_1 r_1 \\ & [\ddot{x}_g \cos(\theta_{c_1} - \theta_1) + \dot{x}_g \dot{\theta}_1 \sin(\theta_{c_1} - \theta_1) + \ddot{y}_g \sin(\theta_{c_1} - \theta_1) - \dot{y}_g \dot{\theta}_1 \cos(\theta_{c_1} - \theta_1)] \\ & + m_2 l' [\ddot{x}_g \cos(\beta' - \theta_1) + \dot{x}_g \dot{\theta}_1 \sin(\beta' - \theta_1) + \ddot{y}_g \sin(\beta' - \theta_1) - \dot{y}_g \dot{\theta}_1 \cos(\beta' - \theta_1)], \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_2} \right) = & m_2 r_2 l \cos \gamma_{2a} \ddot{\theta}_1 - m_2 r_2 l \sin \gamma_{2a} (\dot{\theta}_2 - \dot{\theta}_1) \dot{\theta}_1 + I_{O_2} \ddot{\theta}_2 + m_2 r_2 [\ddot{x}_g \cos(\theta_{c_1} + \theta_2) \\ & - \dot{x}_g \dot{\theta}_2 \sin(\theta_{c_1} + \theta_2) - \ddot{y}_g \sin(\theta_{c_1} + \theta_2) - \dot{y}_g \dot{\theta}_2 \cos(\theta_{c_1} + \theta_2)], \end{aligned}$$

$$\begin{aligned} \frac{\partial T}{\partial \theta_1} = & m_1 r_1 \dot{\theta}_1 [\dot{x}_g \sin(\theta_{c_1} - \theta_1) - \dot{y}_g \cos(\theta_{c_1} - \theta_1)] + m_2 l' \dot{\theta}_1 [\dot{x}_g \sin(\beta' - \theta_1) \\ & - \dot{y}_g \cos(\beta' - \theta_1)] - m_2 r_2 l' \dot{\theta}_1 \dot{\theta}_2 \sin \gamma_{2a}, \end{aligned}$$

$$\frac{\partial T}{\partial \theta_2} = m_2 r_2 \dot{\theta}_2 [-\dot{x}_g \sin(\theta_{c_1} + \theta_2) - \dot{y}_g \cos(\theta_{c_1} + \theta_2)] - m_2 r_2 l' \dot{\theta}_1 \dot{\theta}_2 \sin \gamma_{2a},$$

$$\frac{\partial V}{\partial \theta_1} = m_1 g r_1 \sin(\theta_{c_1} - \theta_1) - m_2 g l' \sin(\beta' - \theta_1), \text{ and}$$

$$\frac{\partial V}{\partial \theta_2} = -m_2 g r_2 \sin(\theta_{c_1} + \theta_2).$$

The governing equations for rocking response in mode 2a are, then, expressed in the form

$$\left. \begin{aligned}
 (I_{O_1} + m_2 l'^2) \ddot{\theta}_1 + m_2 r_2 l' \cos \gamma_{2a} \ddot{\theta}_2 + m_2 r_2 l' \sin \gamma_{2a} \dot{\theta}_2^2 - m_1 g r_1 \sin(\theta_1 - \theta_{c_1}) \\
 - m_2 g l' \sin(\theta_1 - \beta') = -[m_1 r_1 \cos(\theta_1 - \theta_{c_1}) + m_2 l' \cos(\theta_1 - \beta')] \ddot{x}_g \\
 + [m_1 r_1 \sin(\theta_1 - \theta_{c_1}) + m_2 l' \sin(\theta_1 - \beta')] \ddot{y}_g \\
 m_2 r_2 l' \cos \gamma_{2a} \ddot{\theta}_1 + I_{O_2} \ddot{\theta}_2 - m_2 r_2 l' \sin \gamma_{2a} \dot{\theta}_1^2 - m_2 g r_2 \sin(\theta_2 + \theta_{c_2}) \\
 = m_2 r_2 [\ddot{y}_g \sin(\theta_2 + \theta_{c_2}) - \ddot{x}_g \cos(\theta_2 + \theta_{c_2})]
 \end{aligned} \right\} \quad (4.31)$$

which is valid for  $\theta_1 > 0$ ;  $\gamma_{2a}$  is defined by

$$\gamma_{2a} = \theta_1 - \theta_2 - \theta_{c_2} - \beta'. \quad (4.32)$$

For rocking response in mode 2b, the governing equations are similarly derived; they can be written in the form

$$\left. \begin{aligned}
 (I_{O_1} + m_2 l'^2) \ddot{\theta}_1 + m_2 r_2 l' \cos \gamma_{2b} \ddot{\theta}_2 + m_2 r_2 l' \sin \gamma_{2b} \dot{\theta}_2^2 - m_1 g r_1 \sin(\theta_1 + \theta_{c_1}) \\
 - m_2 g l' \sin(\theta_1 + \beta') = -[m_1 r_1 \cos(\theta_1 + \theta_{c_1}) + m_2 l' \cos(\theta_1 + \beta')] \ddot{x}_g \\
 + [m_1 r_1 \sin(\theta_1 + \theta_{c_1}) + m_2 l' \sin(\theta_1 + \beta')] \ddot{y}_g \\
 m_2 r_2 l' \cos \gamma_{2b} \ddot{\theta}_1 + I_{O_2} \ddot{\theta}_2 - m_2 r_2 l' \sin \gamma_{2b} \dot{\theta}_1^2 - m_2 g r_2 \sin(\theta_2 - \theta_{c_2}) \\
 = m_2 r_2 [\ddot{y}_g \sin(\theta_2 - \theta_{c_2}) - \ddot{x}_g \cos(\theta_2 - \theta_{c_2})]
 \end{aligned} \right\} \quad (4.33)$$

which is valid for  $\theta_1 < 0$ . In Equation (4.33)  $\gamma_{2b}$  is defined as

$$\gamma_{2b} = \theta_1 - \theta_2 + \theta_{c_2} + \beta'. \quad (4.34)$$

Equations (4.31) through (4.34) can be combined to give one set of equations for mode 2, namely,

$$\left. \begin{aligned}
& (I_{O_1} + m_2 l^2) \ddot{\theta}_1 + m_2 r_2 l \cos \gamma_2 \ddot{\theta}_2 + m_2 r_2 l \sin \gamma_2 \dot{\theta}_2^2 - m_1 g r_1 \sin(\theta_1 - S_{\theta_1} \theta_{c_1}) \\
& - m_2 g l \sin(\theta_1 - S_{\theta_1} \beta') = -[m_1 r_1 \cos(\theta_1 - S_{\theta_1} \theta_{c_1}) + m_2 l \cos(\theta_1 - S_{\theta_1} \beta')] \ddot{x}_g \\
& + [m_1 r_1 \sin(\theta_1 - S_{\theta_1} \theta_{c_1}) + m_2 l \sin(\theta_1 - S_{\theta_1} \beta')] \ddot{y}_g \\
& m_2 r_2 l \cos \gamma_2 \ddot{\theta}_1 + I_{O_2} \ddot{\theta}_2 - m_2 r_2 l \sin \gamma_2 \dot{\theta}_1^2 - m_2 g r_2 \sin(\theta_2 + S_{\theta_1} \theta_{c_1}) \\
& = m_2 r_2 [\ddot{y}_g \sin(\theta_2 + S_{\theta_1} \theta_{c_1}) - \ddot{x}_g \cos(\theta_2 + S_{\theta_1} \theta_{c_1})]
\end{aligned} \right\} \quad (4.35)$$

where  $S_{\theta_1}$  denotes the signum function in  $\theta_1$ , and

$$\gamma_2 = \theta_1 - \theta_2 - S_{\theta_1}(\theta_{c_1} + \beta'). \quad (4.36)$$

It should be noted that Equations (4.23) and (4.35) are valid provided  $\theta_1 \neq 0$  and  $\theta_1 \neq \theta_2$ ; when  $\theta_1 = 0$  or  $\theta_1 = \theta_2$ , an impact occurs either between block 1 and the ground, or between the two blocks. The relations governing impact are outlined in a general discussion of impact in Chapter 5.

### *Governing Equations for Mode 3*

In mode 3, the two blocks remain in contact ( $\theta_1 = \theta_2$ ), that is, the system rocks as a single rigid body. The equations of motion of the system governing the common angle  $\theta_1$  from the vertical, are derived by considering the equilibrium of moments about the center of rotation. These equations may be written as

$$\left. \begin{aligned}
& I_{O_1} \ddot{\theta}_1 - MRg \sin(\theta_1 - \theta_c) = -MR[\ddot{x}_g \cos(\theta_1 - \theta_c) - \ddot{y}_g \sin(\theta_1 - \theta_c)] \\
& \theta_2 = \theta_1
\end{aligned} \right\} \quad (4.37)$$

which is valid for  $\theta_1 > 0$ , that is, for rotation about  $O_1$  (mode 3a), and

$$\left. \begin{aligned} I_O \ddot{\theta}_1 - MRg \sin(\theta_1 + \theta_c) &= -MR[\ddot{x}_g \cos(\theta_1 + \theta_c) - \ddot{y}_g \sin(\theta_1 + \theta_c)] \\ \theta_2 &= \theta_1 \end{aligned} \right\} \quad (4.38)$$

which is valid for  $\theta_1 < 0$ , that is, for rotation about  $O_1'$  (mode 3b).

In the above equations  $M$  denotes the total mass of the system,  $R$  the distance of the c.g. of the system from any base corner of block 1, and  $I_O$  the mass moment of inertia of the system about any base corner of block 1.

Equations (4.37) and (4.38) govern the motion of the system in mode 3a and mode 3b, respectively. These equations can be combined to yield a compact set of equations for mode 3, namely,

$$\left. \begin{aligned} I_O \ddot{\theta}_1 - MRg \sin(\theta_1 - S_{\theta_1} \theta_c) &= -MR[\ddot{x}_g \cos(\theta_1 - S_{\theta_1} \theta_c) - \ddot{y}_g \sin(\theta_1 - S_{\theta_1} \theta_c)] \\ \theta_2 &= \theta_1 \end{aligned} \right\} \quad (4.39)$$

in which  $S_{\theta_1}$  denotes the signum function in  $\theta_1$ .

#### *Governing Equations for Mode 4*

Mode 4 reflects the rocking of Block 2 alone, with block 1 being still. In this case the equations of motion are

$$\left. \begin{aligned} \theta_1 &= 0 \\ I_{O_2} \ddot{\theta}_2 - m_2 r_2 g \sin(\theta_2 - \theta_{c2}) &= -m_2 r_2 [\ddot{x}_g \cos(\theta_2 - \theta_{c2}) - \ddot{y}_g \sin(\theta_2 - \theta_{c2})] \end{aligned} \right\} \quad (4.40)$$

which is valid for  $\theta_2 > 0$ , that is, for rotation about  $O_2$  (mode 4a), and



$$\left. \begin{aligned} \theta_1 &= 0 \\ I_{O_2} \ddot{\theta}_2 - m_2 r_2 g \sin(\theta_2 + \theta_{c2}) &= -m_2 r_2 [\ddot{x}_g \cos(\theta_2 + \theta_{c2}) - \ddot{y}_g \sin(\theta_2 + \theta_{c2})] \end{aligned} \right\} \quad (4.41)$$

which is valid for  $\theta_2 < 0$ , that is, for rotation about  $O_2'$  (mode 4b).

Equations (4.40) and (4.41) govern the motion of the system in mode 4a and mode 4b, respectively. These equations can be combined to yield a compact set of equations for mode 4, namely,

$$\left. \begin{aligned} \theta_1 &= 0 \\ I_{O_2} \ddot{\theta}_2 - m_2 r_2 g \sin(\theta_2 - S_{\theta_2} \theta_{c2}) &= -m_2 r_2 [\ddot{x}_g \cos(\theta_2 - S_{\theta_2} \theta_{c2}) - \ddot{y}_g \sin(\theta_2 - S_{\theta_2} \theta_{c2})] \end{aligned} \right\} \quad (4.42)$$

in which  $S_{\theta_2}$  denotes the signum function in  $\theta_2$ .

Note that the equations of motion for mode 3 and mode 4, are valid provided  $\theta_1 \neq 0$  and  $\theta_2 \neq 0$ , respectively. When  $\theta_1 = 0$  in mode 3, an impact occurs between the single-block behaving system and the ground. Similarly, when  $\theta_2 = 0$  in mode 4, block 2 impacts the still block 1. The relations governing impact are outlined in a general discussion of impact in Chapter 5.

### 4.3.3 Linearized Equations of Motion

The complexity of the governing equations of motion demonstrate the difficulty of the dynamic analysis of a system of more than one rigid blocks. These equations are highly nonlinear and not amenable to exact closed-form solutions. It should be emphasized that the nonlinear behavior of the system is not only a result of the nonlinear nature of the

equations of motion, but also of the continuous transition between modes each one being governed by a different set of equations.

In order to simplify these equations, the assumption of small angles of rotation is made. Under this assumption, the linearized equations of motion for each mode may be written as

#### Mode 1

$$\left. \begin{aligned} (I_{O_1} + m_2 l^2) \ddot{\theta}_1 + m_2 r_2 l \cos \gamma_{1L} \ddot{\theta}_2 + S_{\theta_1} m_1 r_1 g \sin \theta_{c_1} + S_{\theta_1} m_2 l g \sin \beta &= \\ = -[m_1 r_1 \cos \theta_{c_1} + m_2 l \cos \beta] \ddot{x}_g - S_{\theta_1} [m_1 r_1 \sin \theta_{c_1} + m_2 l \sin \beta] \ddot{y}_g & \\ m_2 r_2 l \cos \gamma_{1L} \ddot{\theta}_1 + I_{O_2} \ddot{\theta}_2 + S_{\theta_2} m_2 g r_2 \sin \theta_{c_2} &= -m_2 r_2 [\ddot{y}_g S_{\theta_2} \sin \theta_{c_2} + \ddot{x}_g \cos \theta_{c_2}] \end{aligned} \right\} \quad (4.43)$$

where  $\gamma_{1L} = S_{\theta_1}(\theta_{c_1} - \beta)$

#### Mode 2

$$\left. \begin{aligned} (I_{O_1} + m_2 l'^2) \ddot{\theta}_1 + m_2 r_2 l' \cos \gamma_{2L} \ddot{\theta}_2 + S_{\theta_1} m_1 g r_1 \sin \theta_{c_1} + S_{\theta_1} m_2 g l' \sin \beta' &= \\ = -[m_1 r_1 \cos \theta_{c_1} + m_2 l' \cos \beta'] \ddot{x}_g - S_{\theta_1} [m_1 r_1 \sin \theta_{c_1} + m_2 l' \sin \beta'] \ddot{y}_g & \\ m_2 r_2 l' \cos \gamma_{2L} \ddot{\theta}_1 + I_{O_2} \ddot{\theta}_2 - S_{\theta_2} m_2 g r_2 \sin \theta_{c_2} &= m_2 r_2 [\ddot{y}_g S_{\theta_2} \sin \theta_{c_2} - \ddot{x}_g \cos \theta_{c_2}] \end{aligned} \right\} \quad (4.44)$$

where  $\gamma_{2L} = -S_{\theta_1}(\theta_{c_1} + \beta')$

#### Mode 3

$$\left. \begin{aligned} I_{O_1} \ddot{\theta}_1 + S_{\theta_1} M R g \sin \theta_c &= -M R [\ddot{x}_g \cos \theta_c + \ddot{y}_g S_{\theta_1} \sin \theta_c] \\ \theta_2 &= \theta_1 \end{aligned} \right\} \quad (4.45)$$

#### Mode 4

$$\left. \begin{aligned} \theta_1 &= 0 \\ I_{O_1} \ddot{\theta}_2 + S_{\theta_1} m_2 r_2 g \sin \theta_{c2} &= -m_2 r_2 [\ddot{x}_g \cos \theta_{c2} + \ddot{y}_g S_{\theta_1} \sin \theta_{c2}] \end{aligned} \right\} \quad (4.46)$$

In the above equations  $S_{\theta_i}$  ( $i = 1, 2$ ) denotes the signum function defined as:  $S_{\theta_i} = 1$  for  $\theta_i > 0$  and  $S_{\theta_i} = -1$  for  $\theta_i < 0$ .

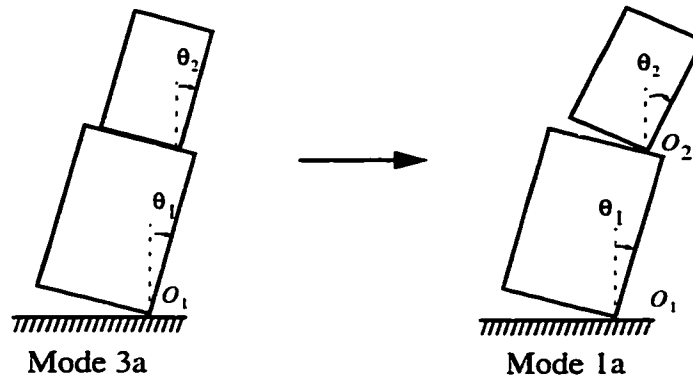
#### 4.3.4 Transition Between Modes without Impact

After the system is set to motion, it continuously switches from one mode to another. Responsible for the transition may either be an impact, or a sudden change in ground excitation. In fact, this transition between modes reflects the highly nonlinear character of the system response. Transition criteria without the occurrence of impact, are outlined below; impact is analyzed extensively in Chapter 5.

Once motion is initiated, this type of transition may occur from mode 3 or mode 4 if the base acceleration is sufficiently strong. The conditions for transition from one mode to another are derived by considering overturning and restoring moments. For example, the system can change from mode 4a to mode 1a if the overturning moment of block 1 about point  $O_1$  becomes greater than the restoring moment.

Transition from mode 3a to mode 1a, Figure 4.5, requires

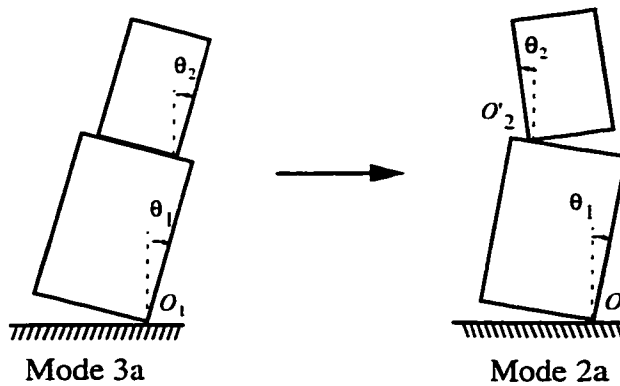
$$\begin{aligned} -\cos(\theta_{c2} - \theta_1) \ddot{x}_g - \sin(\theta_{c2} - \theta_1) \ddot{y}_g &> g \sin(\theta_{c2} - \theta_1) + d \cos(\theta_{c2} + \omega) \dot{\theta}_1^2 \\ &+ \left[ d \sin(\theta_{c2} + \omega) + \frac{I_{G2}}{m_2 r_2} \right] \ddot{\theta}_1. \end{aligned} \quad (4.47)$$



**Fig. 4.5** Transition from Mode 3a to Mode 1a without Impact

Transition from mode 3a to mode 2a, Figure 4.6, requires

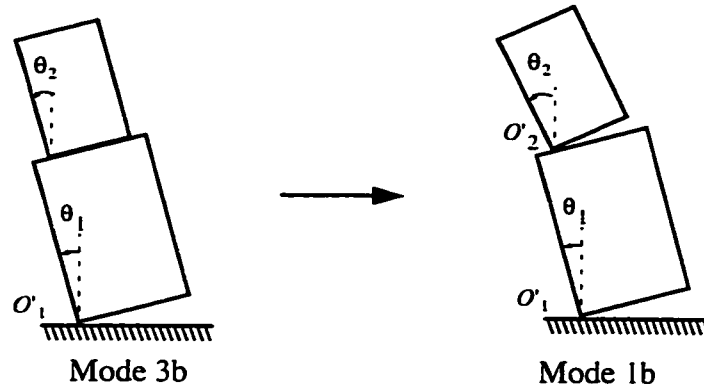
$$\begin{aligned} \cos(\theta_{c2} + \theta_1)\ddot{x}_g - \sin(\theta_{c2} + \theta_1)\ddot{y}_g &> g \sin(\theta_{c2} + \theta_1) + d \cos(2\theta_1 + \theta_{c2} + \omega)\dot{\theta}_1^2 \\ &+ \left[ d \sin(2\theta_1 + \theta_{c2} + \omega) + \frac{I_{G2}}{m_2 r_2} \right] \ddot{\theta}_1. \end{aligned} \quad (4.48)$$



**Fig. 4.6** Transition from Mode 3a to Mode 2a without Impact

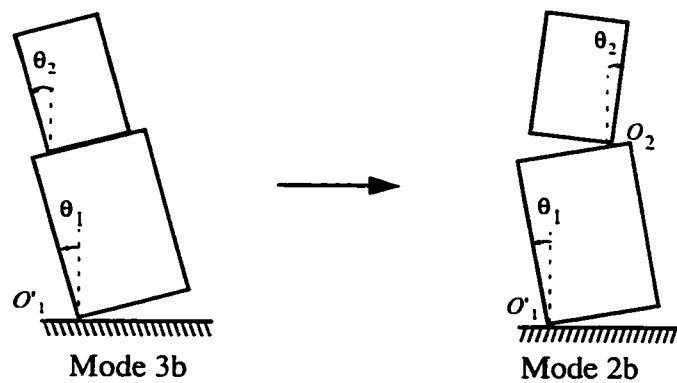
Transition from mode 3b to mode 1b, Figure 4.7, requires

$$\cos(\theta_{c2} + \theta_1)\ddot{x}_g - \sin(\theta_{c2} + \theta_1)\ddot{y}_g > g \sin(\theta_{c2} + \theta_1) + d \sin(\omega - \theta_{c2})\dot{\theta}_1^2 - \left[ d \cos(\theta_{c2} - \omega) - \frac{I_{G2}}{m_2 r_2} \right] \ddot{\theta}_1. \quad (4.49)$$



**Fig. 4.7** Transition from Mode 3b to Mode 1b without Impact

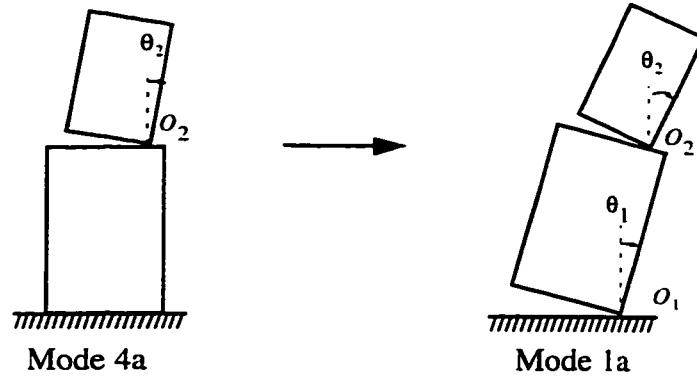
Transition from mode 3b to mode 2b, Figure 4.8, requires



**Fig. 4.8** Transition from Mode 3b to Mode 2b without Impact

$$\begin{aligned}
 -\cos(\theta_{c2} - \theta_1)\ddot{x}_g - \sin(\theta_{c2} - \theta_1)\ddot{y}_g &> g \sin(\theta_{c2} - \theta_1) - d \sin(\omega + \theta_{c2})\dot{\theta}_1^2 \\
 &+ \left[ d \cos(\theta_{c2} + \omega) - \frac{I_{G2}}{m_2 r_2} \right] \ddot{\theta}_1.
 \end{aligned} \quad (4.50)$$

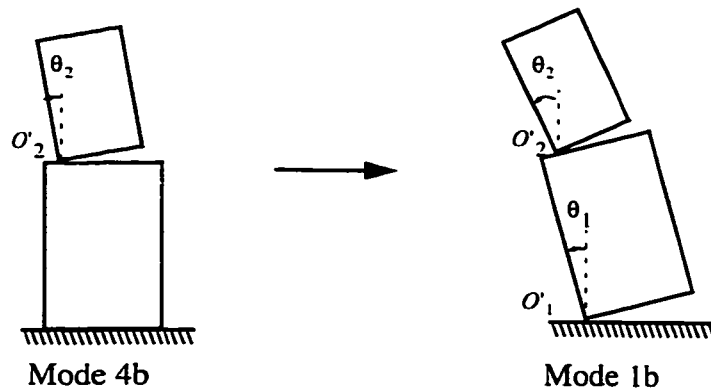
Transition from Mode 4a to Mode 1a, Figure (4.9), requires



**Fig. 4.9** Transition from Mode 4a to Mode 1a without Impact

$$\begin{aligned}
 -(m_1 + 2m_2)h_1\ddot{x}_g - (m_1 b_1 + m_2 \xi)\ddot{y}_g &> m_2 r_2 [\xi \sin(\theta_{c2} - \theta_2) + 2h_1 \cos(\theta_{c2} - \theta_2)]\ddot{\theta}_2 \\
 &+ m_2 r_2 [2h_1 \sin(\theta_{c2} - \theta_2) - \xi \cos(\theta_{c2} - \theta_2)]\dot{\theta}_2^2 + m_1 b_1 g.
 \end{aligned} \quad (4.51)$$

Transition from mode 4b to mode 1b, Figure 4.10, requires

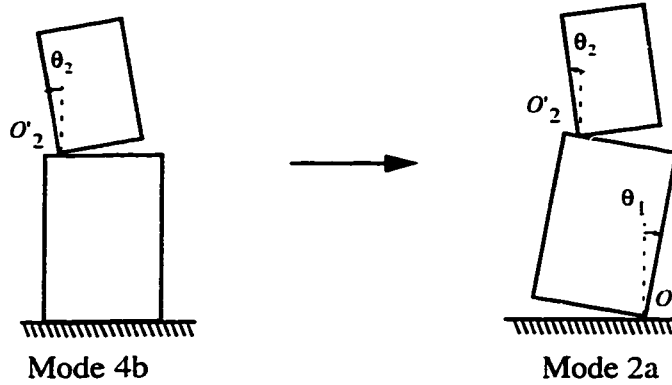


**Fig. 4.10** Transition from Mode 4b to Mode 1b without Impact

$$(m_1 + 2m_2)h_1\ddot{x}_g - (m_1b_1 + m_2\xi)\ddot{y}_g > -m_2r_2[\xi\sin(\theta_{c2} + \theta_2) + 2h_1\cos(\theta_{c2} + \theta_2)]\ddot{\theta}_2 \quad (4.52)$$

$$+ m_2r_2[2h_1\sin(\theta_{c2} + \theta_2) - \xi\cos(\theta_{c2} + \theta_2)]\dot{\theta}_2^2 + m_1b_1g.$$

Transition from mode 4b to mode 2a, Figure 4.11, requires

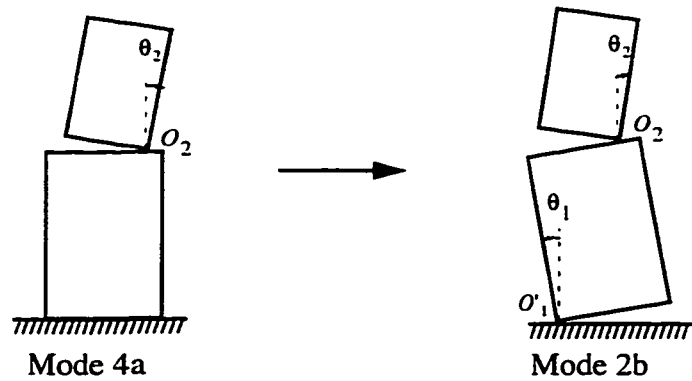


**Fig. 4.11** Transition from Mode 4b to Mode 2a without Impact

$$-(m_1 + 2m_2)h_1\ddot{x}_g - (m_1b_1 + m_2\xi')\ddot{y}_g > -m_2r_2[\xi'\sin(\theta_{c2} + \theta_2) - 2h_1\cos(\theta_{c2} + \theta_2)]\ddot{\theta}_2 \quad (4.53)$$

$$- m_2r_2[2h_1\sin(\theta_{c2} + \theta_2) + \xi'\cos(\theta_{c2} + \theta_2)]\dot{\theta}_2^2 + m_1b_1g.$$

Finally, transition from mode 4a to mode 2b, Figure 4.12, requires



**Fig. 4.12** Transition from Mode 4a to Mode 2b without Impact

$$(m_1 + 2m_2)h_1\ddot{x}_g - (m_1b_1 + m_2\xi')\ddot{y}_g > m_2r_2[\xi'\sin(\theta_{c2} - \theta_2) - 2h_1\cos(\theta_{c2} - \theta_2)]\ddot{\theta}_2 - m_2r_2[2h_1\sin(\theta_{c2} - \theta_2) + \xi'\cos(\theta_{c2} - \theta_2)]\dot{\theta}_2^2 + m_1b_1g. \quad (4.54)$$

In the preceding equations,  $d = \sqrt{(2h_1 + h_2)^2 + b_1^2}$ ,  $\omega = \text{atan}\left(\frac{2h_1 + h_2}{b_1}\right)$ ,  $\xi = b_1 - b_2$ ,

and  $\xi' = 2b_1 - \xi$ .

#### 4.4 Free Vibration

Free “vibration” or free rocking of the system is initiated by rotating the system through an initial angular displacement, say  $\theta_{10} = \theta_{20} \equiv \theta_0$ , releasing it, and letting the system rock back and forth about alternative corners until the motion decays to rest.

Consider the case where the system is set to free rocking by giving a positive initial angular displacement common for the two blocks. The system starts rocking about point  $O_1$  (mode 3a) and remains in this mode until an impact with the foundation occurs. When such an impact takes place, the pole of rotation switches to  $O_1'$  and block 2 may either start rocking separately about  $O_2'$  (mode 1b), or remain in contact with block 1, so that the system continues rocking as one block about  $O_1'$  (mode 3b). From this point on, the system will experience a number of impacts before coming to rest. Note that in free rocking, impact is the only mechanism for transition between modes, with every impact being accompanied by a mode transition.

The equations governing free rocking are derived from Equations (4.23), (4.35), (4.39), and (4.42) with the base acceleration set to zero, namely



*Mode 1*

$$\left. \begin{aligned} (I_{O_1} + m_2 l^2) \ddot{\theta}_1 + m_2 r_2 l \cos \gamma_1 \ddot{\theta}_2 + m_2 r_2 l \sin \gamma_1 \dot{\theta}_2^2 - m_1 r_1 g \sin(\theta_1 - S_{\theta_1} \theta_{c_1}) \\ - m_2 l g \sin(\theta_1 - S_{\theta_1} \beta) = 0 \\ m_2 r_2 l \cos \gamma_1 \ddot{\theta}_1 + I_{O_2} \ddot{\theta}_2 - m_2 r_2 l \sin \gamma_1 \dot{\theta}_1^2 - m_2 g r_2 \sin(\theta_2 - S_{\theta_2} \theta_{c_2}) = 0 \end{aligned} \right\} \quad (4.55)$$

*Mode 2*

$$\left. \begin{aligned} (I_{O_1} + m_2 l^2) \ddot{\theta}_1 + m_2 r_2 l \cos \gamma_2 \ddot{\theta}_2 + m_2 r_2 l \sin \gamma_2 \dot{\theta}_2^2 - m_1 g r_1 \sin(\theta_1 - S_{\theta_1} \theta_{c_1}) \\ - m_2 g l \sin(\theta_1 - S_{\theta_1} \beta') = 0 \\ m_2 r_2 l \cos \gamma_2 \ddot{\theta}_1 + I_{O_2} \ddot{\theta}_2 - m_2 r_2 l \sin \gamma_2 \dot{\theta}_1^2 - m_2 g r_2 \sin(\theta_2 + S_{\theta_2} \theta_{c_2}) = 0 \end{aligned} \right\} \quad (4.56)$$

*Mode 3*

$$\left. \begin{aligned} I_{O_1} \ddot{\theta}_1 - M R g \sin(\theta_1 - S_{\theta_1} \theta_{c_1}) = 0 \\ \theta_2 = \theta_1 \end{aligned} \right\} \quad (4.57)$$

*Mode 4*

$$\left. \begin{aligned} \theta_1 = 0 \\ I_{O_2} \ddot{\theta}_2 - m_2 r_2 g \sin(\theta_{c_2} - S_{\theta_2} \theta_2) = 0 \end{aligned} \right\} \quad (4.58)$$

## CHAPTER 5: The Impact Problem

### 5.1 Introduction

One of the more complicated problems in dynamics is the impact between objects. Impact phenomena involve a fairly complex interrelationship of energy and momentum transfer, energy dissipation, elastic and plastic deformation, relative impact velocity, and body geometry.

The impact of systems of particles is usually treated by a simplified analysis that utilizes the coefficient of restitution - an experimental parameter. This analysis can be extended in the treatment of the impact of rigid bodies - modeled as particles - only in the case of central impact where the contact forces of impact pass through the mass centers of the bodies, as would be the case with colliding smooth spheres, for example. To determine the post-impact velocities of the bodies involved in the impact requires the introduction of the so-called coefficient of restitution  $e$ . Although in the classical theory of impact  $e$  was considered a constant for given materials, more modern investigations show that  $e$  is highly dependent on geometry and impact velocity as well as on materials.

Any attempt to extend this simplified theory of impact utilizing a coefficient of restitution for the noncentral impact of rigid bodies of varying shape greatly oversimplifies the real problem and has little practical value. It should be emphasized that noncentral rigid-body impact is a more complex problem that depends on the geometries of the impacting bodies and their surface characteristics as well as their relative velocities. Herein, the principle of conservation of angular momentum is adopted in treating the impact problem.

Another important factor is the nature of the impact duration. Many impact problems are

amenable to a solution, only if the duration of the impact  $\Delta t$  is so small that the displacement of the bodies during the impact period can be neglected. Any motion that satisfies this condition is called *impulsive motion*. Of course, the results obtained by assuming impulsive motion are only approximations. The analysis is exact only in the idealized case where  $\Delta t \rightarrow 0$ .

## 5.2 Definitions

### *Angular Momentum.*

Consider a system of  $n$  particles. Referring to the arbitrary  $i$ th particle of the system, let  $m_i$  be its mass,  $\mathbf{r}_i$  its position vector measured from a fixed point  $O$ , and  $\mathbf{r}_{i/A}$  its position vector relative to an arbitrary point  $A$ . The angular momentum of the  $i$ th particle about an arbitrary point  $A$  equals the moment of its linear momentum  $\mathbf{p}_i = m_i \mathbf{v}_i$  about  $A$ . The angular momentum of a system of particles is equal to the sum of the angular momenta of all particles in the system, namely,

$$\mathbf{h}_A = \sum_{i=1}^n (\mathbf{h}_A)_i = \sum_{i=1}^n \mathbf{r}_{i/A} \times m_i \mathbf{v}_i. \quad (5.1)$$

A rigid body is assumed to consist of an infinite number of infinitesimal particles, and therefore, Equation (5.1) can be used to derive the angular momentum of a rigid body about an arbitrary point  $A$ . The angular momentum of a rigid body about a point  $A$  (which need neither be fixed nor attached to the body), can be expressed in the form

$$\mathbf{h}_A = \mathbf{h}_G + \mathbf{r}_{G/A} \times m \mathbf{v}_G, \quad (5.2)$$

in which  $\mathbf{v}_G$  is the velocity of the mass center  $G$ ,  $\mathbf{r}_{G/A}$  is the position vector of  $G$  relative

to  $A$ , and  $h_G$  is the angular momentum of the body about its mass center given by

$$h_G = I_G \omega, \quad (5.3)$$

where  $\omega$  is the angular velocity of the body and  $I_G$  is the central mass moment of inertia of the body about an axis perpendicular to the plane of motion.

For plane rigid-body motion, the angular momentum vector is always normal to the plane of motion so that vector notation is generally unnecessary, and Equation (5.2) may be written as

$$h_A = I_G \omega + (mv_G)d \quad (5.4)$$

where  $d$  is the distance of the linear-momentum vector,  $mv_G$ , from point  $A$ .

#### *Moment-angular momentum relationship*

Differentiating the expression for  $h_A$  given in Equation (5.1) with respect to time and noting that the absolute velocity  $v_i$  of the arbitrary  $i$ th particle may be expressed with respect to the velocity of point  $A$  as  $v_i = v_A + \dot{r}_{i/A}$ , one obtains

$$\begin{aligned} \frac{dh_A}{dt} &= \frac{d}{dt} \left\{ \sum_{i=1}^n r_{i/A} \times m_i (v_A + \dot{r}_{i/A}) \right\} \\ &= \sum_{i=1}^n \dot{r}_{i/A} \times m_i (v_A + \dot{r}_{i/A}) + \sum_{i=1}^n r_{i/A} \times m_i \ddot{r}_{i/A}. \end{aligned}$$

But,

$$\sum_{i=1}^n \dot{r}_{i/A} \times m_i v_A = -v_A \times \sum_{i=1}^n m_i \dot{r}_{i/A} = -v_A \times mv_G,$$

and

$$\sum_{i=1}^n \dot{\mathbf{r}}_{i/A} \times m_i \dot{\mathbf{r}}_{i/A} = 0.$$

Therefore,

$$\sum \mathbf{M}_A = \frac{d\mathbf{h}_A}{dt} + \mathbf{v}_A \times m\mathbf{v}_G, \quad (5.5)$$

in which  $\sum \mathbf{M}_A$  is the resultant moment about  $A$  of the forces that are external to the system;  $\mathbf{v}_A$  and  $\mathbf{v}_G$  are the velocities of point  $A$  and system mass center, respectively.

An important special case of this equation is when (a) point  $A$  is fixed, or (b) point  $A$  coincides with the mass center. In such a case, the second term in Equation (5.5) vanishes to yield

$$\sum \mathbf{M}_A = \frac{d\mathbf{h}_A}{dt} \quad (5.6)$$

*Angular impulse-momentum principle.*

Assuming that point  $A$  is fixed or it coincides with the mass center, Equation (5.6) can be integrated over the time interval  $t_1$  to  $t_2$  yielding

$$\int_{t_1}^{t_2} \mathbf{M}_A dt = \int_1^2 d\mathbf{h}_A = \Delta \mathbf{h}_A \quad (5.7)$$

or, “angular impulse equals change in angular momentum” which states the angular impulse-momentum principle. Note that, practically speaking, this principle is valid only if  $A$  is a fixed point or the mass center of a closed system of particles.

### *Conservation of angular momentum*

If the angular impulse about  $A$  is zero, it follows from Equation (5.7) that the angular momentum of the system of particles is conserved about  $A$  :

$$\text{if } \int_{t_1}^{t_2} \mathbf{M}_A dt = 0, \text{ then } (\mathbf{h}_A)_1 = (\mathbf{h}_A)_2 \quad (5.8)$$

where  $(\mathbf{h}_A)_1$  and  $(\mathbf{h}_A)_2$  are the angular momenta about  $A$  at times  $t_1$  and  $t_2$ , respectively.

This is known as the principle of conservation of angular momentum. Note that, the term

$\int_{t_1}^{t_2} \mathbf{M}_A dt$  in the above expression refers to the angular impulse of the external forces.

## **5.3 Analysis of Impact**

### **5.3.1 Preliminaries**

As stated in Section 5.2, the principle of conservation of angular momentum is, in general, valid about the mass center or a fixed point. However, a useful simplification arises in the analysis of rigid-body impact when the motion is assumed to be *impulsive*, meaning that the duration of the impact is negligible. The characteristics of an impulsive motion are summarized below:

*The magnitudes of impact forces tend to infinity:* The impulse of the impact force  $\hat{\mathbf{P}}$  is finite, because it is determined by the impulse-momentum equations. But since  $\hat{\mathbf{P}}$  acts over the infinitesimal time interval  $dt$ , the magnitude of  $\hat{\mathbf{P}}$  must tend to infinity for its impulse to remain finite. A force of infinite magnitude that acts over an infinitesimal time interval and exerts a finite impulse (area under the force-time curve) is called an *impulsive*

*force.* An impulsive force is an example of a Dirac delta function.

*The impulse of finite forces are negligible:* If the magnitude of a force is finite, its impulse during the impact period is negligible. For example, the weight  $W$  of particle is an example of a finite, or nonimpulsive, force. During the time interval  $\Delta t$ , the impulse of  $W$ , i.e.,  $W\Delta t$ , approaches zero as  $\Delta t \rightarrow 0$ .

*The accelerations of the blocks are infinite during impact:* Because the changes in velocities are assumed to occur within an infinitesimal time period, the velocity-time diagrams exhibit jump discontinuities at the time of impact. Since  $a = \frac{dv}{dt}$ , it follows that the ‘jumps’ correspond to infinite accelerations.

*The blocks are in the same location before and after the impact:* Since  $\Delta t \rightarrow 0$ , the distances moved by the blocks during the impact are infinitesimal (the velocities are finite).

Of course, the results obtained by assuming impulsive motion are only approximations. The analysis is exact only in the idealized case where  $\Delta t \rightarrow 0$ . In such a case, the angular momentum about *any* point is conserved. The reason for this simplification is that by assuming the time of impact to be infinitesimal, all displacements during the impact are neglected. Consequently, all points are, in effect, fixed during the impact. A formal proof of this assertion runs as follows.

Recall the moment equation of motion for a system of particles, Equation (5.5) which also applies to a rigid body:

$$\sum M_A = \frac{d\mathbf{h}_A}{dt} + \mathbf{v}_A \times m\mathbf{v}_G.$$

Substituting  $\mathbf{v}_G = \frac{d\mathbf{r}_G}{dt}$  and integrating over the time of impact, one obtains

$$\int_{t_1}^{t_2} \sum \mathbf{M}_A dt = \Delta \mathbf{h}_A + m \mathbf{v}_A \times \int_{t_1}^{t_2} d\mathbf{r}_G, \quad (5.9)$$

where  $\mathbf{r}_G$  is the position vector of the mass center relative to an inertial reference frame.

Since the assumption of impulsive motion implies that the body occupies the same spatial position before, during, and after the impact,  $\mathbf{r}_G$  may be considered to be constant, i.e.,  $d\mathbf{r}_G = 0$ . Therefore, the integral in Equation (5.9) vanishes to yield

$$\int_{t_1}^{t_2} \sum \mathbf{M}_A dt = \Delta \mathbf{h}_A. \quad (5.10)$$

Note that, for impulsive motion, “angular impulse equals change in angular momentum” is valid about any point  $A$ .

If the angular impulse about  $A$  is zero, it follows from Equation (5.10) that the angular momentum of the system is conserved about  $A$ . That is,

$$\text{if } \int_{t_1}^{t_2} \mathbf{M}_A dt = 0, \text{ then } (\mathbf{h}_A)_1 = (\mathbf{h}_A)_2, \quad (5.11)$$

where  $(\mathbf{h}_A)_1$  and  $(\mathbf{h}_A)_2$  are the angular momenta about  $A$  at times  $t_1$  and  $t_2$ , respectively.

Note that, the term  $\int_{t_1}^{t_2} \mathbf{M}_A dt$  in the above expression refers to the angular impulse of the external forces.



### 5.3.2 Formulation

During vibration the system continuously switches from one mode to another when impact between the two blocks or between the first block and the ground occurs. In this study, the foundation is assumed rigid and friction large enough, so that there will be no sliding either of block 1 on the foundation, or of block 2 on block 1. Under the assumption of inelastic impact (no bouncing), the only possible response mechanism is pure rocking about the corners of the two blocks (point-impact). Furthermore, the duration of impact is assumed to be infinitesimal (impulsive motion), implying negligible changes in position and orientation and instantaneous changes in angular velocities. In this regard, the impact is analyzed using the principle of impulse and momentum.

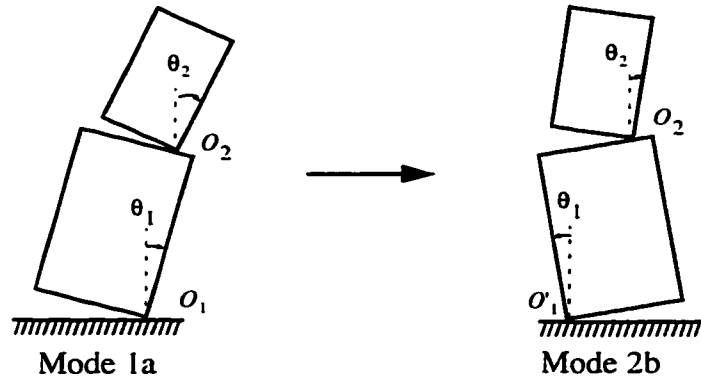
#### *Impact in mode 1*

While in mode 1, the system will in time experience an impact occurring either when the two blocks collide or when block 1 hits the ground. Each case is separately examined below.

##### (a) Block 1 impacts the ground

Assume that the system is rocking in mode 1a. When the angle of rotation of block 1 becomes zero, an impact occurs yielding a change of the rotation pole from  $O_1$  to  $O_1'$ , that is, a switch to mode 2b, Figure 5.1. This results to an instantaneous change in the angular velocities of the blocks, whose post-impact values,  $\dot{\theta}_1^+$  and  $\dot{\theta}_2^+$ , need to be determined. During impact, under the point-impact assumption, the only forces that exert a moment about  $O_1'$  are the weights of the blocks (the impulse forces pass through the point

$O_1'$ ). However, the angular impulse due to these finite forces about  $O_1'$  is extremely small since the time of impact is negligible. Thus, the angular momentum of the system about  $O_1'$  may be said to be conserved in accordance to Equation (5.11).



**Fig. 5.1** Transition from Mode 1a to Mode 2b through Impact

The angular momentum of the system about  $O_1'$  immediately before impact, is simply the sum of angular momenta of each block about the point under consideration, that is,

$$(\vec{h}_{O_1'})_{system} = (\vec{h}_{O_1'})_{block1} + (\vec{h}_{O_1'})_{block2}.$$

Equation (5.4) is, then, used to compute the angular momentum of each block. For example, the angular momentum of block 1 is given by

$$(\vec{h}_{O_1'})_{block1} = I_{G_1} \dot{\theta}_1 + (m_1 \vec{v}_{G_1})d,$$

where  $I_{G_1}$  is the central mass moment of inertia of block 1;  $\dot{\theta}_1$  and  $\vec{v}_{G_1}$  are the angular velocity and the velocity of the mass center for block 1, immediately before impact, and  $d$  is the distance of the linear-momentum vector,  $m\vec{v}_{G_1}$ , from point  $O_1'$ .

The angular momentum of each block, immediately before and after impact, may be similarly computed. Finally, the principle of conservation of angular momentum for the system is expressed in the form

$$(h_{O_1}^-)_{system} = (h_{O_1}^+)_{system}, \quad (5.12)$$

where

$$(h_{O_1}^-)_{sys} = \{I_{O_1} - 2m_1b_1^2 + m_2l[r_2\cos(\theta_2 - \theta_{c2} + \beta) + 2h_1\cos\beta - (2b_1 - \xi)\sin\beta]\}\dot{\theta}_1^- \\ + \{I_{O_2} + m_2r_2[2h_1\cos(\theta_2 - \theta_{c2}) + (2b_1 - \xi)\sin(\theta_2 - \theta_{c2})]\}\dot{\theta}_2^-,$$

and

$$(h_{O_1}^+)_{sys} = \{I_{O_1} + m_2l[r_2\cos(\theta_2 - \theta_{c2} - \beta') + 2h_1\cos\beta' + (2b_1 - \xi)\sin\beta']\}\dot{\theta}_1^+ \\ + \{I_{O_2} + m_2r_2[2h_1\cos(\theta_2 - \theta_{c2}) + (2b_1 - \xi)\sin(\theta_2 - \theta_{c2})]\}\dot{\theta}_2^+.$$

Equation (5.12) involves the two unknown angular velocities,  $\dot{\theta}_1^+$  and  $\dot{\theta}_2^+$ , immediately after impact. A second equation may be derived by considering the conservation of angular momentum about  $O_2$ , for block 2 alone. This is justified by the fact that, since block 2 point-contacts block 1 at  $O_2$ , all impact forces are transmitted to block 2 through that point, so that the angular impulse of block 2 about  $O_2$  is zero (the angular impulse of a finite force like the weight of the block can be neglected). Furthermore since the motion is assumed impulsive, angular impulse equal change in angular momentum, even about the 'moving' point  $O_2$  (See Section 5.3.1). Therefore,

$$(h_{O_2}^-)_{block2} = (h_{O_2}^+)_{block2}, \quad (5.13)$$

where

$$(\dot{h}_{O_1}^-)_2 = [m_2 r_2 l \cos(\theta_2 - \theta_{c2} + \beta)] \dot{\theta}_1^- + I_{O_1} \dot{\theta}_2^-,$$

and

$$(\dot{h}_{O_1}^+)_2 = [m_2 r_2 l \cos(\theta_2 - \theta_{c2} - \beta')] \dot{\theta}_1^+ + I_{O_1} \dot{\theta}_2^+.$$

The system of equations (5.12) and (5.13) is solved for the angular velocities immediately after impact to yield

$$\left. \begin{aligned} \dot{\theta}_1^+ &= A \dot{\theta}_1^- \\ \dot{\theta}_2^+ &= C \dot{\theta}_1^- + \dot{\theta}_2^- \end{aligned} \right\} \quad (5.14)$$

in which

$$A = \frac{A_4 B_1 - A_2 B_3}{A_1 A_4 - A_2 A_3}; \quad C = \frac{A_3 B_1 - A_1 B_3}{A_2 A_3 - A_1 A_4},$$

where

$$A_1 = I_{O_1} + m_2 l^2 [r_2 \cos(\theta_2 - \theta_{c2} - \beta') + 2h_1 \cos \beta' + (2b_1 - \xi) \sin \beta'],$$

$$A_2 = I_{O_1} + m_2 r_2 [2h_1 \cos(\theta_2 - \theta_{c2}) + (2b_1 - \xi) \sin(\theta_2 - \theta_{c2})],$$

$$A_3 = m_2 r_2 l \cos(\theta_2 - \theta_{c2} - \beta'),$$

$$A_4 = I_{O_1},$$

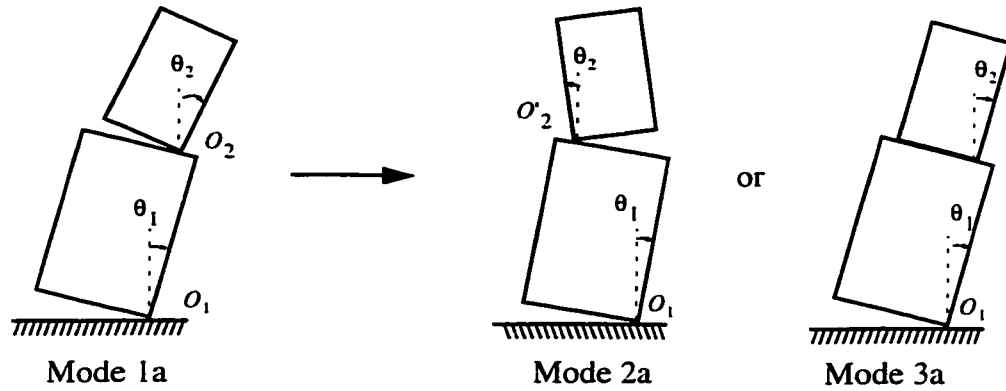
$$B_1 = I_{O_1} - 2m_1 b_1^2 + m_2 l^2 [r_2 \cos(\theta_2 - \theta_{c2} + \beta) + 2h_1 \cos \beta - (2b_1 - \xi) \sin \beta], \text{ and}$$

$$B_3 = m_2 r_2 l \cos(\theta_2 - \theta_{c2} + \beta).$$

Note that, from symmetry, the same analysis also holds for the case of transition from mode 1b to mode 2a.

(b) Impact between the blocks

Consider the case where the system is rocking in mode 1a. If the angle of rotation of block 2 becomes equal to that of block 1, an impact between the blocks occurs and block 2 may either start rocking about  $O_2'$  (mode 2a), or remain in contact with block 1, implying that the system rocks as one rigid-body about  $O_1$  (mode 3a), Figure 5.2.



**Fig. 5.2** Transition from Mode 1a to Mode 2a or Mode 3a through Impact

Assume that the system switches from mode 1a to mode 2a. Considering the system, it is seen that the forces generated during impact are applied at the point  $O_1$ , so that the angular impulse of the system about that point is zero. Consequently, the angular momentum of the system about  $O_1$  is conserved during the time of impact, namely,

$$(\dot{h}_{O_1})_{system} = (h_{O_1}^+)_{system}, \quad (5.15)$$

where

$$(\dot{h}_{O_1}^-)_{system} = \{I_{O_1} + m_2 l \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\zeta - \beta)\} \dot{\theta}_1^- \\ + \{I_{G_2} + m_2 r_2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\theta_2 - \theta_1 - \theta_{c2} + \zeta)\} \dot{\theta}_2^-,$$

and

$$(\dot{h}_{O_1}^+)_{system} = \{I_{O_1} + m_2 l \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\zeta - \beta')\} \dot{\theta}_1^+ \\ + \{I_{G_2} + m_2 r_2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\theta_2 - \theta_1 + \theta_{c2} + \zeta)\} \dot{\theta}_2^+.$$

A second equation may be derived by considering the conservation of angular momentum about  $O_2'$ , for the block 2 alone. Since impact takes place at  $O_2'$ , and neglecting the angular impulse of the weight of the block, the angular impulse of block 2 about that point is zero. Furthermore, since the motion is assumed impulsive, the principle of conservation of angular momentum is still valid about the 'non-stationary' point  $O_2'$ . Therefore,

$$(\dot{h}_{O_2'}^-)_{block2} = (\dot{h}_{O_2'}^+)_{block2}, \quad (5.16)$$

where

$$(\dot{h}_{O_2'}^-)_{block2} = m_2 r_2 l \cos(\theta_2 - \theta_1 + \theta_{c2} + \beta) \dot{\theta}_1^- + I_{G_2} + m_2 r_2^2 \cos(2\theta_{c2}) \dot{\theta}_2^-,$$

and

$$(\dot{h}_{O_2'}^+)_{block2} = m_2 r_2 l' \cos(\theta_2 - \theta_1 + \theta_{c2} + \beta') \dot{\theta}_1^- + I_{O_2'} \dot{\theta}_2^-.$$

The system of equations (5.15) and (5.16) is solved for the angular velocities immediately after impact to derive

$$\left. \begin{aligned} \dot{\theta}_1^+ &= A\dot{\theta}_1^- + B\dot{\theta}_2^- \\ \dot{\theta}_2^+ &= C\dot{\theta}_1^- + D\dot{\theta}_2^- \end{aligned} \right\} \quad (5.17)$$

in which

$$A = \frac{A_4 B_1 - A_2 B_3}{A_1 A_4 - A_2 A_3}; \quad B = \frac{A_4 B_2 - A_2 B_4}{A_1 A_4 - A_2 A_3};$$

$$C = \frac{A_3 B_1 - A_1 B_3}{A_2 A_3 - A_1 A_4}; \quad D = \frac{A_3 B_2 - A_1 B_4}{A_2 A_3 - A_1 A_4};$$

where

$$A_1 = I_{O_1} + m_2 l' \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\zeta - \beta'),$$

$$A_2 = I_{G_2} + m_2 r_2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\theta_2 - \theta_1 + \theta_{c2} + \zeta),$$

$$A_3 = m_2 r_2 l' \cos(\theta_2 - \theta_1 + \theta_{c2} + \beta'),$$

$$A_4 = I_{O_2},$$

$$B_1 = I_{O_1} + m_2 l \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\zeta - \beta),$$

$$B_2 = I_{G_2} + m_2 r_2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\theta_2 - \theta_1 - \theta_{c2} + \zeta),$$

$$B_3 = m_2 r_2 l \cos(\theta_2 - \theta_1 + \theta_{c2} + \beta), \text{ and}$$

$$B_4 = I_{G_2} + m_2 r_2^2 \cos(2\theta_{c2}).$$

This analysis was carried out assuming that after impact, block 1 and block 2 rock sepa-

rately about  $O_1$  and  $O_2'$ , respectively (mode 2a). If, however, it turns out that  $\dot{\theta}_2^+ > \dot{\theta}_1^+$ , which is physically impossible, then after impact, block 2 remains in contact with block 1 and the system rocks as one rigid body about  $O_1$  (mode 3a). In such a case,  $\dot{\theta}_1^+ = \dot{\theta}_2^+$ , and its value is derived by considering the conservation of angular momentum about  $O_1$  for the whole system. That is,

$$(h_{O_1}^-)_{system} = (h_{O_1}^+)_{system}, \quad (5.18)$$

where

$$\begin{aligned} (h_{O_1}^-)_{system} = & \{ I_{O_1} + m_2 l \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\zeta - \beta) \} \dot{\theta}_1^- \\ & + \{ I_{G_2} + m_2 r_2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\theta_2 - \theta_1 - \theta_{c2} + \zeta) \} \dot{\theta}_2^-, \end{aligned}$$

and

$$(h_{O_1}^+)_{system} = I_O \dot{\theta}_1^+,$$

in which  $I_O = I_{O_1} + I_{G_2} + m_2 [b_1^2 + (2h_1 + h_2)^2]$  is the mass moment of inertia of the system about  $O_1$ .

The final relation is

$$\dot{\theta}_1^+ = \dot{\theta}_2^+ = \left( \frac{B_1}{I_O} \right) \dot{\theta}_1^- + \left( \frac{B_2}{I_O} \right) \dot{\theta}_2^-, \quad (5.19)$$

where

$$B_1 = I_{O_1} + m_2 l \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\zeta - \beta),$$



and

$$B_2 = I_{G_2} + m_2 r_2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \cos(\theta_2 - \theta_1 - \theta_{c2} + \zeta).$$

The same analysis also holds for the case of transition from mode 1b to mode 2b or 3b.

### ***Impact in mode 2***

Let the system rock in mode 2 when an impact occurs. The impact analysis is again subdivided into two cases depending on whether impact occurs between the two blocks or between the first block and the foundation. Each case is separately examined below.

#### **(a) Block 1 impacts the ground**

If the system is rocking in mode 2a when block 1 impacts the foundation, transition to mode 1b occurs, Figure 5.3. In order to find the angular velocities after impact, the conservation of angular momentum about  $O_1'$  for the system, and about  $O_2'$  for the upper block is employed. The equation concerning the conservation of angular momentum about  $O_1'$  for the system is

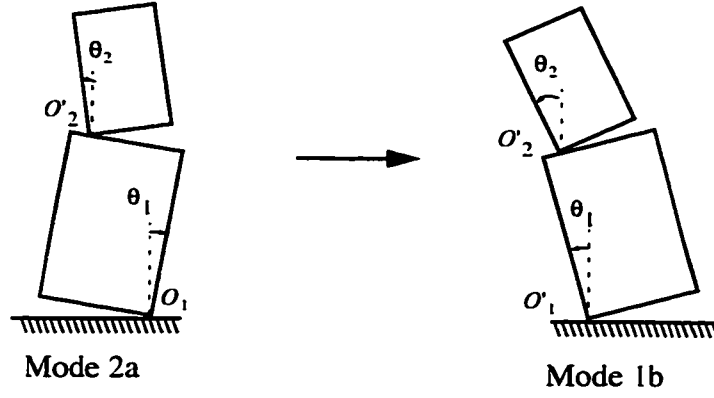
$$(h^{-}_{O_1'})_{system} = (h^{+}_{O_1'})_{system}, \quad (5.20)$$

in which

$$\begin{aligned} (h^{-}_{O_1'})_{system} = & \{I_{O_1} - 2m_1 b_1^2 + m_2 l' [r_2 \cos(\theta_2 + \theta_{c2} + \beta') + 2h_1 \cos \beta' - \xi \sin \beta']\} \dot{\theta}_1 \\ & + \{I_{O_2} + m_2 r_2 [2h_1 \cos(\theta_2 + \theta_{c2}) + \xi \sin(\theta_2 + \theta_{c2})]\} \dot{\theta}_2, \end{aligned}$$

and

$$(h^{+}_{O_1'})_{system} = \{I_{O_1} + m_2 l [r_2 \cos(\theta_2 + \theta_{c2} - \beta) + 2h_1 \cos \beta + \xi \sin \beta]\} \dot{\theta}_1^{+} \\ + \{I_{O_1} + m_2 r_2 [2h_1 \cos(\theta_2 + \theta_{c2}) + \xi \sin(\theta_2 + \theta_{c2})]\} \dot{\theta}_2^{+}.$$



**Fig. 5.3** Transition from Mode 2a to Mode 1b through Impact

The conservation of angular momentum about  $O_2'$  for the upper block is expressed in the form

$$(h^{-}_{O_2'})_{block2} = (h^{+}_{O_2'})_{block2}, \quad (5.21)$$

where

$$(h^{-}_{O_2'})_{block2} = m_2 r_2 l' \cos(\theta_2 + \theta_{c2} + \beta') \dot{\theta}_1^{-} + I_{O_2'} \dot{\theta}_2^{-},$$

and

$$(h^{+}_{O_2'})_{block2} = m_2 r_2 l \cos(\theta_2 + \theta_{c2} - \beta) \dot{\theta}_1^{+} + I_{O_2'} \dot{\theta}_2^{+}.$$

Equations (5.20) and (5.21) are simultaneously solved for the angular velocities immediately after impact to yield

$$\left. \begin{aligned} \dot{\theta}_1^+ &= A\dot{\theta}_1^- + B\dot{\theta}_2^- \\ \dot{\theta}_2^+ &= C\dot{\theta}_1^- + D\dot{\theta}_2^- \end{aligned} \right\} \quad (5.22)$$

where

$$A = \frac{A_4 B_1 - A_2 B_3}{A_1 A_4 - A_2 A_3}; \quad B = \frac{A_4 B_2 - A_2 B_4}{A_1 A_4 - A_2 A_3};$$

$$C = \frac{A_3 B_1 - A_1 B_3}{A_2 A_3 - A_1 A_4}; \quad D = \frac{A_3 B_2 - A_1 B_4}{A_2 A_3 - A_1 A_4};$$

in which

$$A_1 = I_{O_1} + m_2 l [r_2 \cos(\theta_2 + \theta_{c2} - \beta) + 2h_1 \cos \beta + \xi \sin \beta],$$

$$A_2 = I_{O_1} + m_2 r_2 [2h_1 \cos(\theta_2 + \theta_{c2}) + \xi \sin(\theta_2 + \theta_{c2})],$$

$$A_3 = m_2 r_2 l \cos(\theta_2 + \theta_{c2} - \beta),$$

$$A_4 = I_{O_1},$$

$$B_1 = I_{O_1} - 2m_1 b_1^2 + m_2 l [r_2 \cos(\theta_2 + \theta_{c2} + \beta') + 2h_1 \cos \beta' - \xi \sin \beta'],$$

$$B_2 = I_{O_1} + m_2 r_2 [2h_1 \cos(\theta_2 + \theta_{c2}) + \xi \sin(\theta_2 + \theta_{c2})],$$

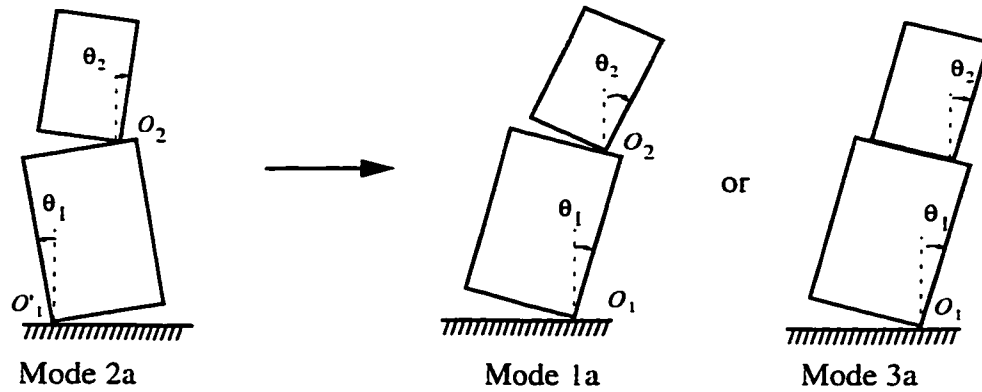
$$B_3 = m_2 r_2 l \cos(\theta_2 + \theta_{c2} + \beta'), \text{ and}$$

$$B_4 = I_{O_1}.$$

The above analysis is also valid for transition from mode 2b to mode 1a.

## (b) Impact between the blocks

Assume that the system is rocking in mode 2a. If the angle of rotation of block 2 becomes equal to that of block 1, an impact between the blocks occurs and block 2 tends to start rocking about  $O_2$  (mode 1a). It is also possible that, after impact, the two blocks remain in full contact, so that the system rocks as one rigid-body about  $O_1$  (mode 3a), Figure 5.4.



**Fig. 5.4** Transition from Mode 2a to Mode 1a or to Mode 3a through Impact

Consider the case where the system switches from mode 2a to mode 1a. In order to find the angular velocities after impact, the conservation of angular momentum about  $O_1$  for the whole system, and about  $O_2$  for the upper block is employed, namely

$$(\dot{h}_{O_1}^-)_{system} = (\dot{h}_{O_1}^+)_{system}, \quad (5.23)$$

where

$$\begin{aligned} (\dot{h}_{O_1}^-)_{system} = & \{ I_{O_1} + m_2 l^2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \sin(\theta_1 + \zeta - \beta') \} \dot{\theta}_1^- \\ & + \{ I_{G_2} + m_2 r_2^2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \sin(\theta_2 + \theta_{c2} + \zeta) \} \dot{\theta}_2^-, \end{aligned}$$

and

$$(h_{O_i}^+)_{system} = \{I_{O_i} + m_2 l \sqrt{b_1^2 + (2h_1 + h_2)^2} \sin(\theta_1 + \zeta - \beta)\} \dot{\theta}_1^+ + I_{O_i} \dot{\theta}_2^+.$$

Similarly,

$$(h_{O_i}^-)_{block2} = (h_{O_i}^+)_{block2}, \quad (5.24)$$

where

$$(h_{O_i}^-)_{block2} = m_2 r_2 l \cos(\theta_1 - \theta_2 + \theta_{c2} - \beta') \dot{\theta}_1^- + [I_{G_i} + m_2 r_2^2 \cos(\theta_1 + \theta_2 - \theta_{c2} - \beta') \dot{\theta}_2^-],$$

and

$$(h_{O_i}^+)_{block2} = m_2 r_2 l \cos(\theta_1 - \theta_2 + \theta_{c2} - \beta) \dot{\theta}_1^+ + I_{O_i} \dot{\theta}_2^+.$$

The system of equations (5.23) and (5.24) is solved for the angular velocities immediately after impact. The final results are

$$\left. \begin{aligned} \dot{\theta}_1^+ &= A \dot{\theta}_1^- + B \dot{\theta}_2^- \\ \dot{\theta}_2^+ &= C \dot{\theta}_1^- + D \dot{\theta}_2^- \end{aligned} \right\} \quad (5.25)$$

where

$$A = \frac{A_4 B_1 - A_2 B_3}{A_1 A_4 - A_2 A_3}; \quad B = \frac{A_4 B_2 - A_2 B_4}{A_1 A_4 - A_2 A_3};$$

$$C = \frac{A_3 B_1 - A_1 B_3}{A_2 A_3 - A_1 A_4}; \quad D = \frac{A_3 B_2 - A_1 B_4}{A_2 A_3 - A_1 A_4};$$

in which

$$A_1 = I_{O_1} + m_2 l \sqrt{b_1^2 + (2h_1 + h_2)^2} \sin(\theta_1 + \zeta - \beta),$$

$$A_2 = I_{O_1},$$

$$A_3 = m_2 r_2 l \cos(\theta_1 - \theta_2 + \theta_{c2} - \beta),$$

$$A_4 = I_{O_1},$$

$$B_1 = I_{O_1} + m_2 l \sqrt{b_1^2 + (2h_1 + h_2)^2} \sin(\theta_1 + \zeta - \beta'),$$

$$B_2 = I_{G_2} + m_2 r_2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \sin(\theta_2 + \theta_{c2} + \zeta),$$

$$B_3 = m_2 r_2 l \cos(\theta_1 - \theta_2 + \theta_{c2} - \beta'), \text{ and}$$

$$B_4 = I_{G_2} + m_2 r_2^2 \cos(\theta_1 + \theta_2 - \theta_{c2} - \beta').$$

The preceding analysis assumed that after impact block 1 and block 2 rock separately about  $O_1$  and  $O_2$ , respectively (mode 1a). If, however, this analysis leads to  $\dot{\theta}_2^+ < \dot{\theta}_1^+$ , which is not physically realizable, then after impact, block 2 remains in contact with block 1 and the system rocks as one rigid body about  $O_1$  (mode 3a). In such a case,  $\dot{\theta}_1^+ = \dot{\theta}_2^+$ , and its value is derived by considering the conservation of angular momentum about  $O_1$  for the whole system. That is,

$$(\dot{h}_{O_1})_{system} = (h_{O_1}^+)_{system}, \quad (5.26)$$

where

$$\begin{aligned} (\bar{h}_{O_1})_{system} = & \{I_{O_1} + m_2 l^2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \sin(\theta_1 + \zeta - \beta')\} \dot{\theta}_1 \\ & + \{I_{G_2} + m_2 r_2^2 \sqrt{b_1^2 + (2h_1 + h_2)^2} \sin(\theta_2 + \theta_{c2} + \zeta)\} \dot{\theta}_2, \end{aligned}$$

and

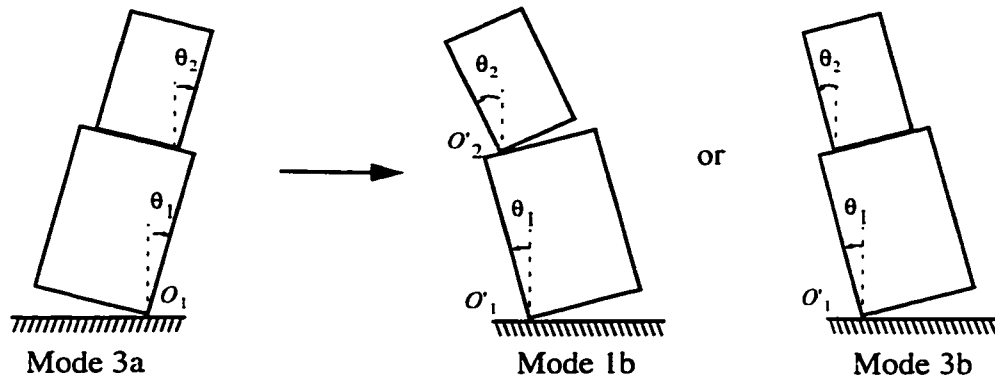
$$(h_{O_1}^+)_{system} = \left(\frac{B_1}{I_O}\right) \dot{\theta}_1 + \left(\frac{B_2}{I_O}\right) \dot{\theta}_2;$$

$B_1$  and  $B_2$  as defined in Equation (5.25).

The same analysis also holds for transition from mode 2b to mode 1b or to mode 3b.

### **Impact in mode 3**

Consider the case where the two blocks are rocking together as one block about point  $O_1$  (mode 3a). If impact with the foundation occurs, the pole of rotation switches to  $O_1'$  and block 2 may either start rocking independently about  $O_2'$  (mode 1b), or remain in contact with block 1, so that the system continues rocking as one block about  $O_1'$  (mode 3b), Figure 5.4.



**Fig. 5.5** Transition from Mode 3a to Mode 1b or Mode 3b through Impact

Consider first the case where the system switches from mode 3a to mode 1b. Evaluation of the angular velocities after impact can be carried out by employing the conservation of angular momentum about  $O_1'$  for the system, and about  $O_2'$  for block 2.

The conservation of angular momentum for the system may be expressed in the form

$$(h_{O_1'}^-)_{system} = (h_{O_1'}^+)_{system}, \quad (5.27)$$

where

$$(h_{O_1'}^-)_{system} = \{I_O - 2Mb_1^2\}\dot{\theta}_1^-,$$

and

$$\begin{aligned} (h_{O_1'}^+)_{system} = & \{I_{O_1} + m_2l[(2h_1 + h_2)\cos\beta + b_1\sin\beta]\}\dot{\theta}_1^+ \\ & + \{I_{G_2} + m_2r_2[b_1\sin\theta_{c2} + (2h_1 + h_2)\cos\theta_{c2}]\}\dot{\theta}_2^+. \end{aligned}$$

The second equation, comes from the conservation of angular momentum of block 2 about  $O_2'$ , namely,

$$(h_{O_2'}^-)_{block2} = (h_{O_2'}^+)_{block2}, \quad (5.28)$$

where

$$(h_{O_2'}^-)_{block2} = \{I_{O_1} + m_2l[h_2\cos\beta' - b_2\sin\beta']\}\dot{\theta}_1^-,$$

and

$$(h_{O_2'}^+)_{block2} = m_2l[h_2\cos\beta + b_2\sin\beta]\dot{\theta}_1^+ + I_{O_2'}\dot{\theta}_2^+.$$

The system of equations (5.27) and (5.28) is solved for the angular velocities immediately



after impact to yield

$$\left. \begin{aligned} \dot{\theta}_1^+ &= A\dot{\theta}_1^- \\ \dot{\theta}_2^+ &= C\dot{\theta}_1^- \end{aligned} \right\} \quad (5.29)$$

where

$$A = \frac{A_4 B_1 - A_2 B_2}{A_1 A_4 - A_2 A_3}; \quad C = \frac{A_3 B_1 - A_1 B_2}{A_2 A_3 - A_1 A_4};$$

in which

$$A_1 = I_{O_1} + m_2 l [(2h_1 + h_2) \cos \beta + b_1 \sin \beta],$$

$$A_2 = I_{G_1} + m_2 r_2 [b_1 \sin \theta_{c2} + (2h_1 + h_2) \cos \theta_{c2}],$$

$$A_3 = m_2 l [h_2 \cos (\beta + b_2 \sin \beta)],$$

$$A_4 = I_{O_1},$$

$$B_1 = I_O - 2M b_1^2, \text{ and}$$

$$B_2 = I_{O_1} + m_2 l' [h_2 \cos \beta' - b_2 \sin \beta'].$$

This analysis was carried out assuming that after impact block 1 and block 2 rock separately about  $O_1'$  and  $O_2'$ , respectively (mode 1b). If, however, it turns out that  $\dot{\theta}_2^+ > \dot{\theta}_1^+$ , which is physically impossible, then after impact, block 2 remains in contact with block 1 and the system rocks as one block about  $O_1'$  (mode 3b). In such a case,  $\dot{\theta}_1^+ = \dot{\theta}_2^+$ , and its value is derived by considering the conservation of angular momentum about  $O_1'$  for the

system. The result is

$$(h_{O_1'}^-)_{system} = (h_{O_1'}^+)_{system}, \quad (5.30)$$

where

$$(h_{O_1'}^-)_{system} = (I_O - 2Mb_1^2)\dot{\theta}_1^-,$$

as in Equation (5.27), and

$$(h_{O_1'}^+)_{system} = I_O\dot{\theta}_1^+.$$

Equation (5.30) is solved for the common post-impact angular velocity giving

$$\dot{\theta}_1^+ = \dot{\theta}_2^+ = \left( \frac{I_O - 2Mb_1^2}{I_O} \right) \dot{\theta}_1^-. \quad (5.31)$$

The above relations also hold for transition from mode 3b to mode 1a or to mode 3a.

#### ***Impact in mode 4***

Assume that the system is rocking in mode 4a. If the angle of rotation of block 2 becomes equal to zero, an impact with block 1 takes place and the pole of rotation of block 2 switches to  $O_2'$ . Furthermore, block 1 may either start rocking about  $O_1'$  (mode 1b), or remain still, so that only rocking of block 2 is happening (mode 3b), Figure 5.6.

Consider the case where the system switches from mode 4a to mode 1b. In order to find the angular velocities after impact, the principle of conservation of angular momentum about  $O_1'$  for the system, and about  $O_2'$  for block 2 is employed. Considering the system, one obtains

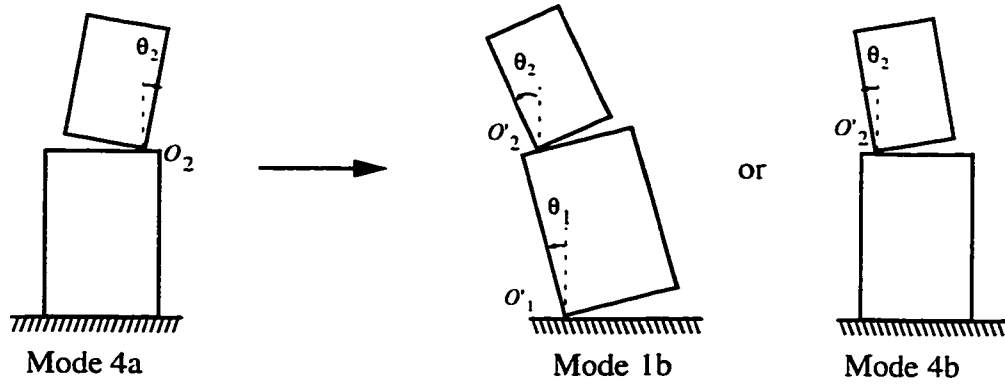
$$(h_{O_1'}^-)_{system} = (h_{O_1'}^+)_{system}, \quad (5.32)$$

where

$$(h_{O_1'}^-)_{system} = \{I_{G_2} + m_2 r_2 [(2h_1 + h_2) \cos \theta_{c2} - b_1 \sin \theta_{c2}]\} \dot{\theta}_2^-,$$

and

$$(h_{O_1'}^+)_{system} = \{I_{O_1} + m_2 l [(2h_1 + h_2) \cos \beta + b_1 \sin \beta]\} \dot{\theta}_1^+ + \{I_{G_2} + m_2 r_2 [(2h_1 + h_2) \cos \theta_{c2} + b_1 \sin \theta_{c2}]\} \dot{\theta}_2^+.$$



**Fig. 5.6** Transition from Mode 4a to Mode 1b or Mode 4b through Impact

Conservation of angular momentum about  $O_2'$  for block 2, gives

$$(h_{O_2'}^-)_{block2} = (h_{O_2'}^+)_{block2}, \quad (5.33)$$

where

$$(h_{O_2'}^-)_{block2} = \{I_{O_2} - 2m_2 b_2^2\} \dot{\theta}_2^-,$$

and

$$(h_{O_1}')_{block2} = m_2 l [h_2 \cos \beta + b_2 \sin \beta] \dot{\theta}_1^+ + I_{O_1} \dot{\theta}_2^+.$$

The system of equations (5.32) and (5.33) is solved for the angular velocities immediately after impact to yield

$$\left. \begin{aligned} \dot{\theta}_1^+ &= B \dot{\theta}_2^- \\ \dot{\theta}_2^+ &= D \dot{\theta}_2^- \end{aligned} \right\} \quad (5.34)$$

in which

$$B = \frac{A_4 B_1 - A_2 B_2}{A_1 A_4 - A_2 A_3}; \quad D = \frac{A_3 B_1 - A_1 B_2}{A_2 A_3 - A_1 A_4}$$

where

$$A_1 = I_{O_1} + m_2 l [(2h_1 + h_2) \cos \beta + b_1 \sin \beta],$$

$$A_2 = I_{G_1} + m_2 r_2 [(2h_1 + h_2) \cos \theta_{c2} + b_1 \sin \theta_{c2}],$$

$$A_3 = m_2 l [h_2 \cos \beta + b_2 \sin \beta],$$

$$A_4 = I_{O_1},$$

$$B_1 = I_{G_1} + m_2 r_2 [(2h_1 + h_2) \cos \theta_{c2} - b_1 \sin \theta_{c2}], \text{ and}$$

$$B_2 = I_{O_1} - 2m_2 b_2^2.$$

This analysis was carried out assuming that after impact block 1 and block 2 rock separately about  $O_1'$  and  $O_2'$ , respectively (mode 1b). If, however, it turns out that  $\dot{\theta}_1^+ > 0$ , which is physically impossible, then after impact, block 1 continues to remain still and

only block 2 is rocking (mode 4b). In such a case,  $\dot{\theta}_2^+$  is derived by considering the conservation of angular momentum about  $O_2'$  for block 2, yielding

$$(h_{O_2'}^-)_{block2} = (h_{O_2'}^+)_{block2} , \quad (5.35)$$

where

$$(h_{O_2'}^-)_{block2} = \{I_{O_2'} - 2m_2 b_2^2\} \dot{\theta}_2^- ,$$

as in Equation (5.33), and

$$(h_{O_2'}^+)_{block2} = I_{O_2'} \dot{\theta}_2^+ .$$

Equation (5.35) is solved for the post-impact angular velocity to give

$$\dot{\theta}_2^+ = \left( \frac{I_{O_2'} - 2m_2 b_2^2}{I_{O_2'}} \right) \dot{\theta}_2^- . \quad (5.36)$$

The same analysis also holds for the case of transition from mode 4b to mode 1a or to mode 4a.

## CHAPTER 6: Numerical Solution

### 6.1 Preliminaries

The equations of motion for all modes of response were derived in Section 4.3. These equations are highly nonlinear and not amenable to exact closed-form solutions even if the time variation of the excitation is described by a simple analytic function. Numerical methods are therefore essential in deriving an approximate solution for the response of such a nonlinear system. In this study, a fourth-order Runge-Kutta method was employed. A brief discussion of its basis is presented here.

Problems involving ordinary differential equations can always be reduced to the study of sets of first order differential equations via a state-space formulation. Therefore, the fundamental problem in the numerical solution of ordinary differential equations is the solution of the first-order equation

$$\frac{dy(t)}{dt} = f(t, y(t)) \quad (6.1)$$

subject to the initial condition  $y = y_0$  when  $t = t_0$ .

Since  $y$  as an explicit function of  $t$  does not exist, the objective is to obtain satisfactory approximations to the values of the solution  $y(t)$  on a specified set of  $t$  values,  $t_1, t_2, \dots, t_N$ . The fourth-order Runge-Kutta method, approximates the exact solution  $y(t)$  by the stepwise iteration algorithm

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (6.2)$$

in which

$$\begin{aligned}
k_1 &= hf(t_i, y_i) \\
k_2 &= hf\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right) \\
k_4 &= hf(t_i + h, y_i + hk_3)
\end{aligned} \tag{6.3}$$

where  $h$  denotes the constant incremental step in time.

The method demonstrated above for solving Equation (6.1) can be extended to systems involving differential equations of order higher than one, since an  $n$ th-order equation reduces to a set of  $n$  1st-order equations.

The classical fourth-order Runge-Kutta algorithm is by far the most popular method for obtaining numerical solutions to differential equations. It is relatively easy to code, and in general, very stable and accurate.

## 6.2 Structure of the Program

A computer program has been developed for calculating numerically the response of two stacked rigid bodies to base excitation. The program solves for the response under general conditions, considering all four modes of response, large angles of rotation, impact, and arbitrary base excitation. The exact highly nonlinear equations of motion are integrated for each mode, and any potential transition between modes is automatically accounted for.

The program is unique in that it considers the many different modes of response. It determines the correct mode of response, integrates the corresponding equations of motion and checks the results to determine if they are consistent with the given mode. If this is not the case, a mode transition occurs. Responsible for the transition may either be an impact,

which is governed by the impact model derived in Chapter 5, or a sudden change in ground excitation.

The program has been written in Matlab. Integration of the governing equations of motion is accomplished using a fourth-order Runge-Kutta algorithm. For this reason, a state-space formulation is employed to yield a system of four first-order differential equations. It is to be noted that, great effort was devoted to developing the algorithms to track the response and correctly handle the transition phenomena, rather to integrating the equations as an isolated problem.

A typical calculation runs as follows. Upon reading the input parameters, the program is initialized and the initial mode of response is determined by the motion initiation criteria given in Section 4.3.1. The equations of motion corresponding to that mode are then integrated within the time interval from  $t_1=0$  to  $t_2=t_1+h$ , in which  $h$  is the user-specified time step. The computed response is then checked to determine if a mode change has occurred. If not, the appropriate equations of motion are integrated from  $t_2$  to  $t_3=t_2+h$  and the process is repeated.

Suppose that at some arbitrary time  $t_i$  the system is rocking in mode 1a ( $\theta_2 > \theta_1 > 0$ ). If the response at the next step  $t_{i+1}$ , which is computed using the equation of motion for mode 1a, yields a negative angle of rotation for block 1 ( $\theta_1 < 0$ ), then these results are not valid. This implies that an impact between the first block and the foundation has occurred ( $\theta_1=0$ ) yielding a mode change to mode 2b. Then, taking the angular velocities reported at time  $t_i$  as the pre-impact angular velocities, the post-impact velocities are evaluated



according to the equations governing impact in Chapter 5. These values are then used as initial values for the equations governing the next mode of vibration.

A similar procedure is carried out for eight different mode conditions encountered. These conditions are associated with a potential mode transition or overturning of the system. Note that, although impact is always accompanied by a mode transition, a mode change can also occur as a result of a sudden change in ground excitation. The criterion for overturning of the system, checked after each time step, refers to overturning of any of the two blocks and is defined herein by  $\theta_i = \pi/2$ . It should be emphasized that  $\theta_i = \theta_{ci}$  cannot be used as criterion for overturning, since the loading under consideration is of dynamic - not static - nature.

### 6.3 Applications

The dynamic response of the system for the case of free rocking and under ground excitation has been computed for various system parameters. It should be emphasized, though, that the aim of these calculations was not to perform an exhaustive parameter study, rather to simply demonstrate the application of the program.

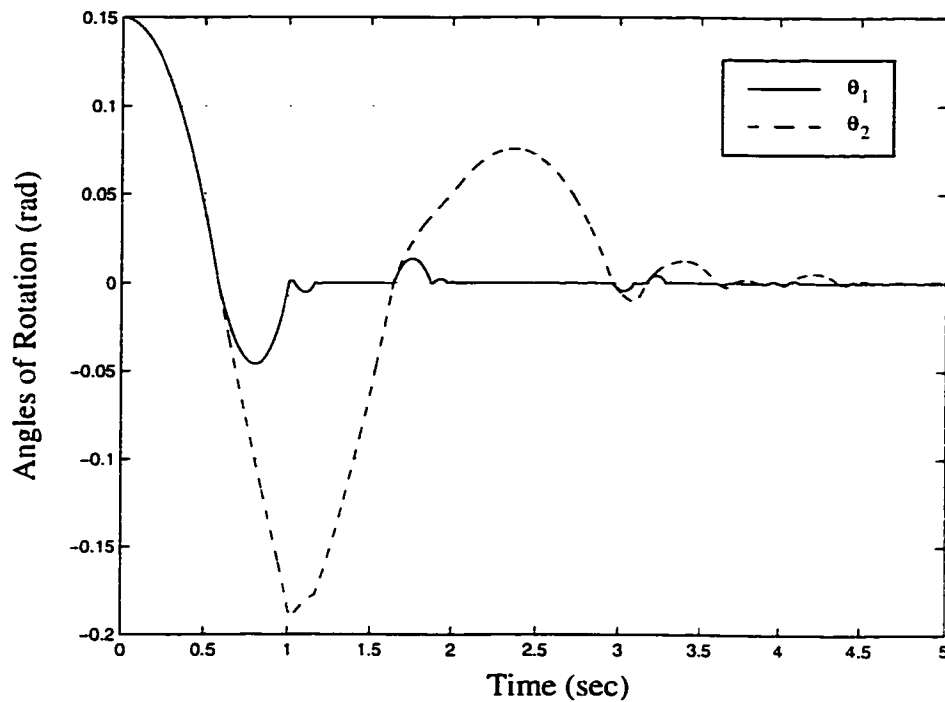
#### 6.3.1 Free Vibration Response

Free “vibration” or free rocking of the system is initiated by rotating the system through an initial angular displacement, say  $\theta_{10} = \theta_{20} \equiv \theta_0$ , releasing it, and letting the system rock back and forth about alternative corners until the motion decays to rest.

In the following examples, a system of two blocks having slenderness ratios  $H_1/B_1 = 1$  and  $H_2/B_2 = 2.5$  with base width  $B_1 = 1.25$  m and  $B_2 = 1.0$  m were used. The blocks were

assumed homogeneous of equal material densities  $\rho_1=\rho_2=2500\text{kg}/\text{m}^3$ . The nonlinear equations of motion were integrated using a fourth order Runge-Kutta method with time step of 0.001 sec.

Presented in Figure 6.1 are results for  $\theta_{10}=\theta_{20}=0.15\text{rad}$  and zero initial angular velocities. It is seen that the angles of rotation of the two blocks are decaying in an oscillatory manner with that of block 1 decreasing at a faster rate.



**Fig. 6.1** Free Vibration Response of a Two-Block System  
( $B_1=H_1=1.25$  m;  $B_2=1.0$  m,  $H_2=2.5$  m).

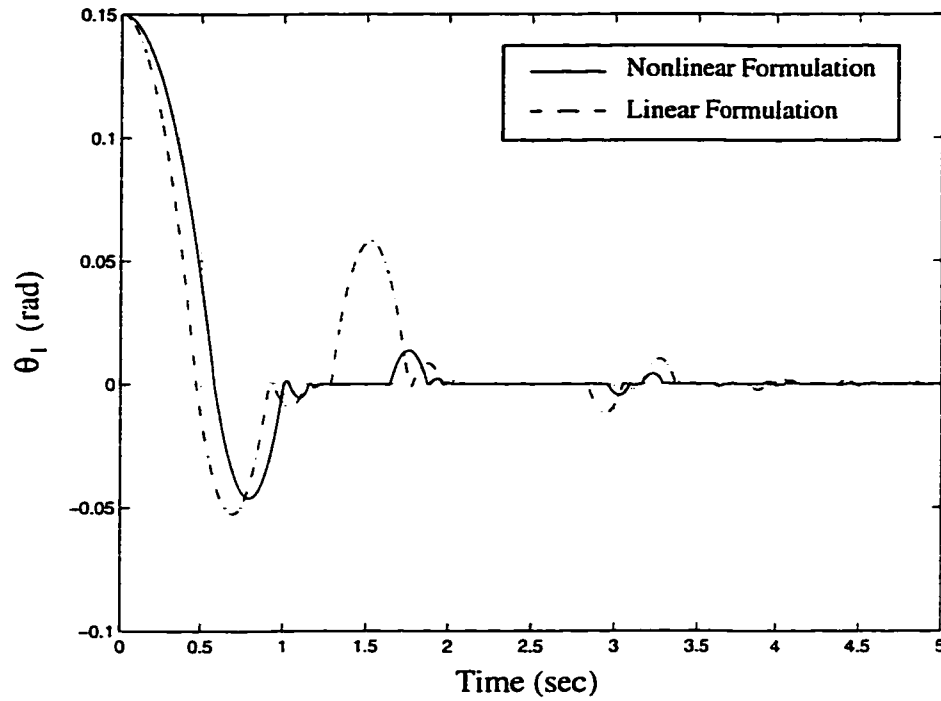
The system starts rocking about point  $O_1$  (mode 3a) and remains in this mode until an impact with the foundation occurs. When such an impact takes place, the pole of rotation switches to  $O_1'$  and block 2 starts rocking independently about  $O_2'$  (mode 1b). From this

point on, the system will experience a number of impacts before coming to rest. Note that in free rocking, impact is the only mechanism for transition between modes, with every impact being accompanied by a mode transition.

In this example, block 1 comes temporarily practically to rest very quickly, since the amplitude of the second cycle is already very small. This block will in time experience a few more short periods of rocking, generally as a result of an impact with block 2. For the cases where block 1 can be thought of as being still, the system may be assumed to rock in mode 4. In contrast, block 2 performs a number of cycles before coming to rest, with decreasing amplitude in each cycle, except for the second half cycle. At  $t \approx 1$  sec, the angle of rotation reaches a value of 0.183 rad exceeding the initial value  $\theta_{20} = 0.15$  rad. This is not surprising, since the angular velocity of block 2 increases during the first impact. Note also that, the period of vibration significantly changes in each cycle for both blocks.

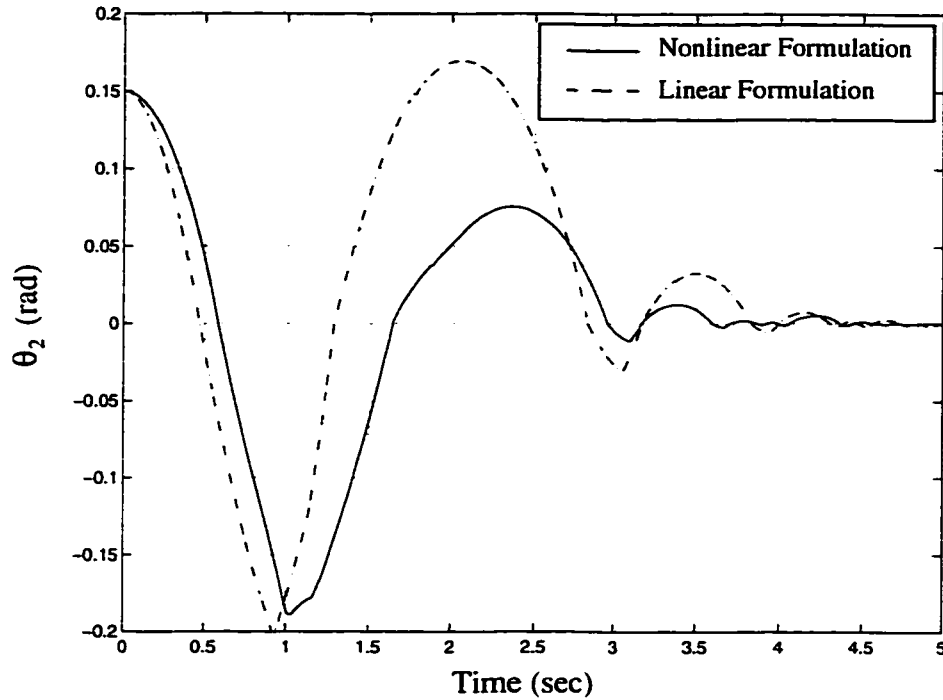
For the single rocking block, Housner (1963) derived a relation between the amplitude of vibration and the period. Apparently, for the two-block system, such a correlation is not easy to derive, due to the complexity of the phenomenon. It can be said, however that, in general, larger amplitudes correspond to larger periods, although the mode of vibration plays an important role as well.

As a second example, the response of the system under the conditions mentioned above, was also calculated numerically employing the linearized equations derived in Section 4.3.3 for small angles of rotation. The results for both the linear and nonlinear formulation are illustrated in Figures 6.2 and 6.3.



**Fig. 6.2** Rotation Angle for Block 1: Nonlinear vs. Linear Formulation

Referring to these figures, it can be seen that the linear formulation yields an increased system response. Furthermore, this amplification seems to be affecting more the intermediate stages of motion. Attention should also be drawn to the fact that there is a time lack in the two responses.

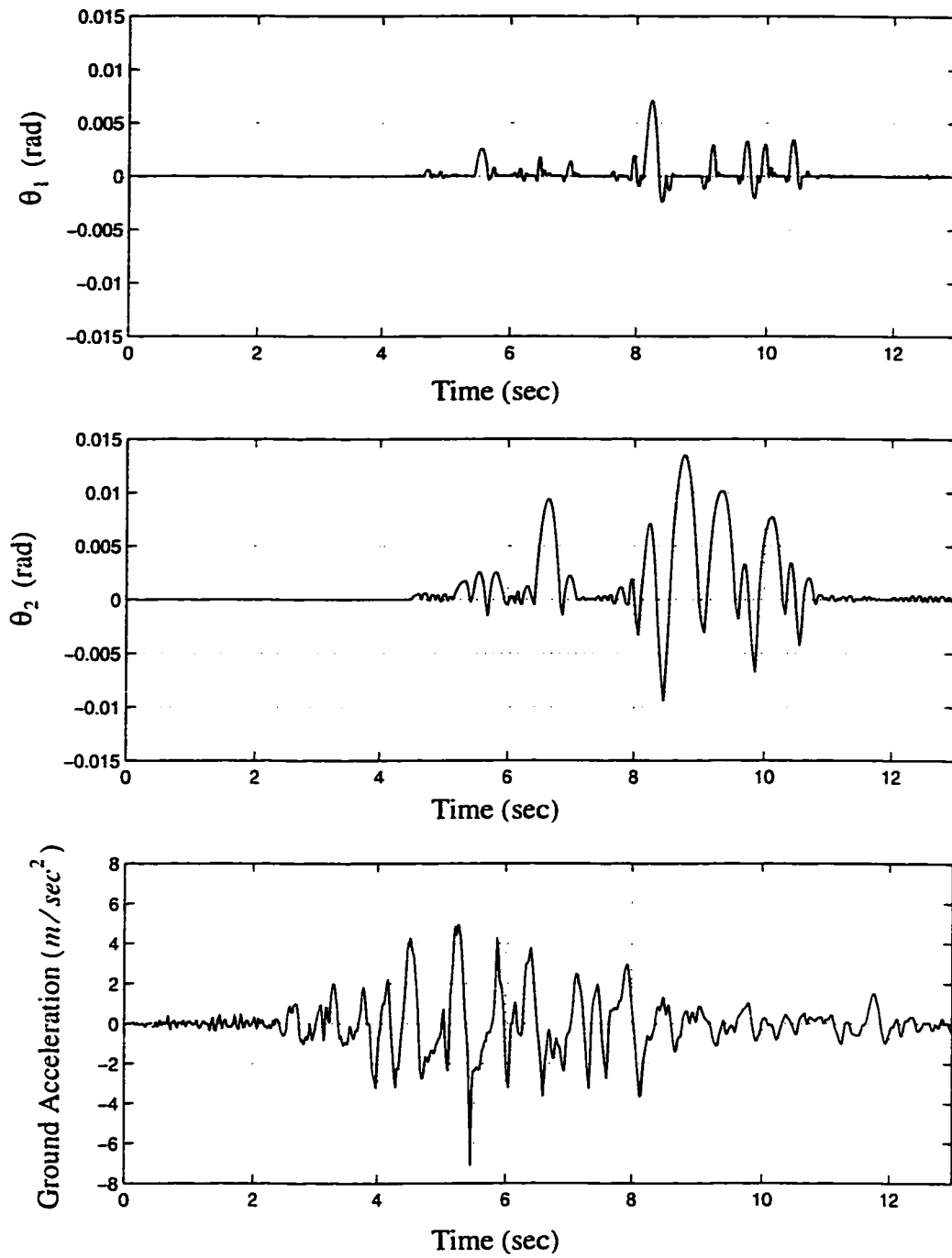


**Fig. 6.3** Rotation Angle for Block 2: Nonlinear vs. Linear Formulation

### 6.3.2 Response to Earthquake Excitation

As a second case, the response of the system to a horizontal earthquake excitation was computed. Results are presented here for an accelerogram recorded at Simi Valley during the Northridge CA, 1994, earthquake. This record is shown in Figure 6.4c and has a peak amplitude of 0.71g. The blocks were assumed homogeneous of equal material densities  $\rho_1 = \rho_2 = 2500 \text{ kg/m}^3$  with geometric characteristics:  $B_1 = H_1 = 1.25 \text{ m}$ ,  $B_2 = 1.0 \text{ m}$ , and  $H_2 = 2.5 \text{ m}$ . The first 13 sec of the response of the system are shown in Figure 6.4.

Referring to Figure 6.4, it is observed that the response of block 1 is, in general, much less than that of block 2, in amplitude and duration. In particular, block 1 is set into motion whenever a large peak on the excitation occurs, or when it is hit by block 2, in which case



**Fig. 6.4** Response to Earthquake Ground Motion  
 ( $B_1 = 1.25m$ ,  $H_1 = 2.5m$ ;  $B_2 = 1.0m$ ,  $H_2 = 2.5m$ )

the system rocks as one rigid block (mode 3a or 3b). Attention should also be drawn to the fact that the maximum response of the two blocks, which occurs at different times, does not immediately follow the peak amplitude in the ground acceleration. Finally, the system comes to rest before the end of the earthquake record.

It should be emphasized that the behavior of the system is highly nonlinear and small changes in the system parameters may alter the response significantly. Therefore, the foregoing quantitative results, cannot be generalized.

## CHAPTER 7: Conclusions

The dynamic behavior of structures of two stacked rigid blocks subjected to ground excitation is examined. Assuming no sliding, the rocking response of the system standing free on a rigid foundation is investigated. In this problem, the most probable failure mechanism is the loss of equilibrium of the whole structure or part of it.

The analytical formulation of this nonlinear problem proved quite challenging. Its complexity is associated with the continuous transition from one regime of motion (mode) to another, each one being governed by a different set of highly nonlinear equations. Furthermore, depending on the mode, this transition can happen either through impact, occurring between the blocks or between the lower block and the ground, or without impact by an instantaneous separation of the two blocks.

The exact (nonlinear) equations governing the rocking response of the system to horizontal and vertical ground acceleration are derived for each mode. Furthermore, a model governing impact is derived using the principle of impulse and momentum. Finally, a computer program is developed to solve for the response of the system under base excitation. The program solves for the response under general conditions, considering all four modes of response, large angles of rotation, impact, and arbitrary base excitation. The full nonlinear equations of motion are integrated for each mode using a fourth-order Runge-Kutta integration scheme, and any potential transition between modes is automatically accounted for. To illustrate the application of the program, the response of the system in free rocking and under ground excitation has been computed. It should be emphasized that the behavior of the system is highly nonlinear and the system response is poorly condi-



tioned; small changes in input or geometr may yield large changes in system response. Therefore, the quantitative results obtained in the application examples cannot be generalized.

The current model may be extended to incorporate some of the characteristics of a non-rigid foundation. For example, by inserting an elastic layer between the lower block and the rigid ground, the complex impact problem may be avoided. Furthermore, it is possible to undertake a comprehensive numerical investigation into the generalized response for realistic ground motions, utilizing the computer program developed in this work. Using an ensemble of synthetic earthquake ground motions, response statistics may be obtained for a range of system parameters, from which possible systematic trends in the system response may be identified. Experimental investigation of the problem is also expected in future work. This attempt is essential in verifying the numerical solution of the nonlinear equations of motion. Finally, a dynamic analysis of multi-block structures, consisting of more than two blocks, might be attempted with a method analogous to the one described in this work.

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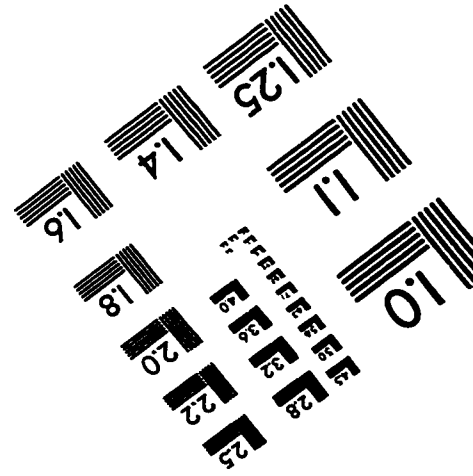
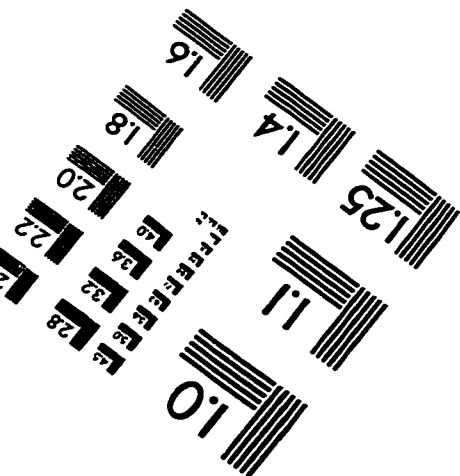
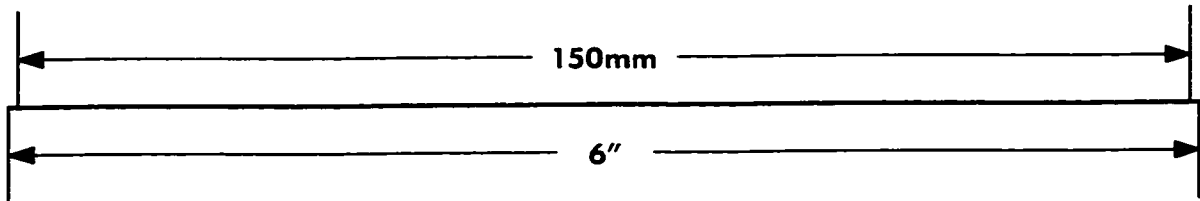
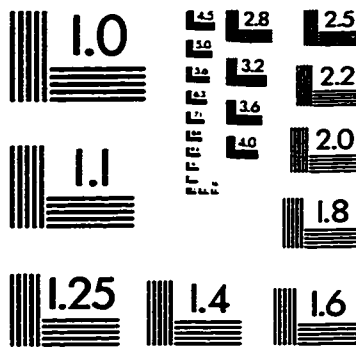
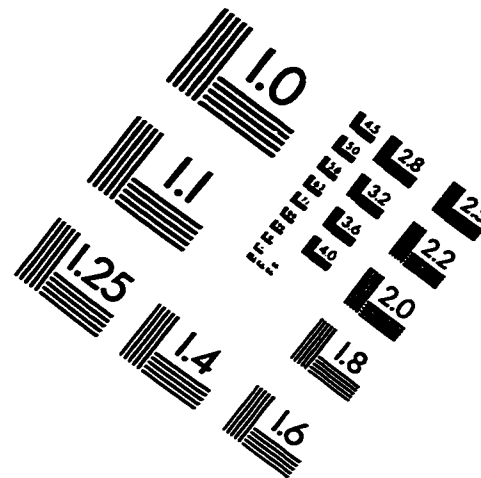
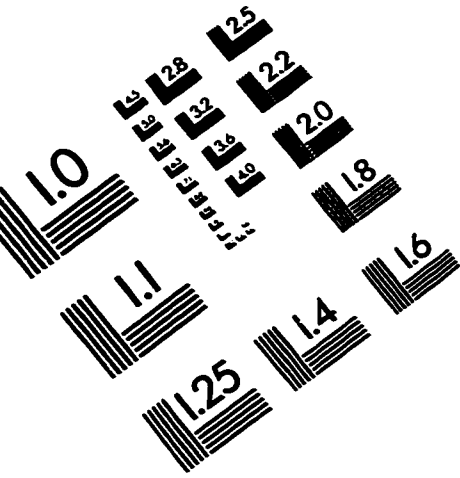
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