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THE FOURIER COEFFICIENTS OF SIEGEL MODULAR FORMS OF DEGREE TWO

by

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Abstract

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In this thesis, two conjectures concerning the Fourier coefficients of Siegel modular forms of degree two are presented. From these two conjectures, which have a certain "naturality" and simplicity within the framework of known results, are derived formulae which completely determine the generator of the graded ring of modular forms of even weight through its Fourier coefficients. Additionally, to add credence to the conjectures, one of three known methods of generating Fourier coefficients of modular forms is used to obtain a table of coefficients with which to illustrate the conjectures. It may be mentioned that this set of Fourier coefficients, in itself, represents the first known table of any length for the Siegel modular forms of degree two.

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Part Two-Table of Fourier Coefficients

Introduction

One of the main results of a paper by H. Resnikoff [7] is that the graded ring (finite sums) of Siegel modular forms of degree two is generated by the Eisenstein series ϕ_4. Consequently, this has given rise to a new interest in the form ϕ_4 and has, in fact, led the author to two conjectures concerning the Fourier coefficients of these modular forms. From these two conjectures can be derived new formulae which will completely determine ϕ_4 through its Fourier coefficients.

The purpose of this paper is to present these two conjectures and the derived results. Additionally, to add credence to the conjectures, we have chosen one of three known methods of generating Fourier coefficients of modular forms and have obtained a table of coefficients with which to illustrate the conjectures. This set of Fourier coefficients, in itself, represents the first known table of any length for the Siegel modular forms of degree two.

Background: Investigations into the Fourier coefficients of Siegel modular forms began with the work of C. L. Siegel [9,10] in 1939 in which the coefficients for the Eisenstein series of degree n were expressed in terms of p-adic densities. His formula, by its generality, is extra-ordinarily complex and proves to be unapplicable for extensive calculations. In 1964 for the degree two Eisenstein series, H. Maass [6] expanded the p-adic densities of Siegel's formula to obtain an explicit expression for the Fourier coefficients associated with primitive matrices. This formula is of course still complicated.

What Maass needed to complete the evaluation of the Eisenstein coefficients in the degree two case was an equation for the imprimitive matrix coefficients. Using an extention of the Hecke operator theory, Maass [6] determined an identity satisfied by the Eisenstein coefficients. He then attempted to show that the identity could be simplified to yield a recursive equation for the imprimitive matrix coefficients and that the coefficients, $a_w(T)$, for $T=\begin{pmatrix} t_1 & t_3/2 \\ t_3/2 & t_2 \end{pmatrix}$ were a function only of the two parameters $e(T)=\text{g.c.d.}(t_1,t_2,t_3)$ and $D(T)=|2T|/e^2(T)$. Unfortunately, we have found an error in his proof ([6]-Satz 2) which invalidates his simplified recursive equation and leaves the dependence of

$a_w(T)$ on $e(T)$ and $D(T)$ an open question.

In Appendix I of this paper, we will be able to correct the error in Maass' proof to show that the Hecke operator identity is indeed recursive for the imprimitive matrix coefficients. However, we will not be able to simplify the identity nor be able to conclude that $a_w(T)$ is a function only of $e(T)$ and $D(T)$. This being the case, we single out this last statement as Maass' Conjecture.

In 1970 using certain differential operators which are independent of the modular group, Resnikoff showed his result that ϕ_4 generates the graded ring of Siegel modular forms of even weight. Later he pointed out the existence of recursive equations for the Fourier coefficients of the four algebraic generators of this same graded ring which could be obtained from the differential operator theory [8]. These equations while complicated are not intertwined with the number theoretic complexities of Siegel's and Maass' formulae. Moreover, they demonstrate how one generates other modular forms from the knowledge of the ϕ_4-coefficients. For this reason, we have chosen this method for generating the table in this paper(p.79).

<u>The Conjectures:</u> In this thesis, we state two conjectures concerning the Fourier coefficients of modular forms of degree two. It may be mentioned that these two conjectures have a certain naturality and simplicity within the framework of known results. Moreover, for the large set of coefficients generated by the method above, the conjectures are found to hold.

The first conjecture is an explicit expression for the Fourier coefficients associated with imprimitive matrices in terms of primitive matrix coefficients. Thus once the primitive matrix coefficients are determined, one also has the imprimitive matrix coefficients. The equation of the conjecture has a form similar to the known solution of the degree one Eisenstein series coefficients, and appears to hold for all modular forms of arbitrary weights. An immediate consequence of the conjecture is a trivial proof of Maass' Conjecture.

In contrast to this, the second conjecture applies only to the Fourier coefficients of the Eisenstein series of weight four. This conjecture relates a sum of these degree two ϕ_4-coefficients to the known coefficients of the classical (degree one) Eisenstein series of weight four. A geometric similarity between this conjecture and an identity of E. Witt is readily noticed.

Finally, under the assumption of the second conjecture, we prove recursive equations for the ϕ_4-coefficients of primitive matrices. Coupling this with a special case of the first conjecture, we have the formulae to completely determine ϕ_4. This points out that if the conjectures are true, then there still exist some unknown fundamental relationships between the classical modular forms and forms of higher degree.

The first two chapters, comprising the first part of this paper, are devoted to development and to the two conjectures. Chapter I considers some general facts pertaining to modular forms in n-space; while Chapter II contains the two conjectures and the derived consequences. The last three chapters (Part Two) take up the subject of generating a table of Fourier coefficients. The recursive equations and as much of the differential theory as is necessary to derive them are given in Chapter III. Chapter IV explores some aspects of evaluating the coefficients by the equations and outlines an algorithm used on a Control Data Corporation 3800 computer. The table of obtained coefficients is given in Chapter V along with an extended table for the ϕ_4 coefficients calculated by means of the equations derived from the conjectures.

For comparisions, Siegel's generating formula as well as Maass' results are given in Appendix I. Also included is our proof of the recursive nature of the Hecke operator identity mentioned above. Appendix II contains the computer time and cost requirements needed to complete the table in Part Two of this paper.

PART ONE

PRELIMINARIES AND THE TWO CONJECTURES

I. General Remarks

Prior to stating and deriving consequences of the conjectures of this paper, it will be necessary to formalize some of the concepts to be used. Since the conjectures will relate modular forms of degrees 1 and 2, we have found it reasonable to present most of the material in this chapter in its general setting for degree n. Included are results which we can call upon, as the occasions arise, later in this paper.

A. Modular Forms

Let H_n^+ denote the set of all symmetric complex n-by-n matrices, $Z = Z^{(n)} = X + iY = (z_{k\ell})$ $(1 \le k \le \ell \le n)$, with positive definite imaginary part Y. In symbols, this means $Z = Z^T$ and $Y > 0$, where $(\cdot)^T$ indicates matrix transpose and $Y > 0$ indicates that the matrix Y is positive definite. By definition, H_n^+ is the Siegel Upper-Half Plane of degree n. The (homogenous) Modular Group of degree n is the group $M_{(2n)}$ consisting of all 2n-by-2n matrices M satisfying the condition

$$\text{(I-1)} \qquad M\,I\,M^T = I; \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \; I = \begin{pmatrix} O & E \\ -E & O \end{pmatrix}$$

where A,B,C,D are integral square matrices of order n and $E = E^{(n)}$ and $0 = 0^{(n)}$ are respectively the identity and zero matrices of order n.

$M_{(2n)}$ has in H_n^+ a discontinuous analytic representation given by the mapping

$$Z \to M\langle Z\rangle = (AZ+B)\cdot(CZ+D)^{-1}$$

of H_n^+ onto itself. Since the matrices M and -M give the same mapping, we define the <u>Siegel Modular Group</u> of degree n, Γ_n, to be the factor group of $M_{(2n)}$ by its normal subgroup $\{E^{(2n)}, -E^{(2n)}\}$ of order two.

<u>Definition 1</u>: A complex-valued function f(Z) on H_n^+ is said to be a <u>Siegel modular form</u> of degree n and weight w (denoted briefly as $f\in(\Gamma_n,w)$) if

(i) f(Z) is an holomorphic function of the $\frac{n(n+1)}{2}$ complex varibles $z_{k\ell}$ $(1\le k\le \ell\le n)$ of Z in H_n^+,

(ii) For every $M = \begin{pmatrix} AB \\ CD \end{pmatrix}$,

$$f(M\langle Z\rangle)\ |CZ+D|^{-w} = f(Z)$$

where $|\cdot|$ indicates determinant.

It has been shown by Koecher [3] that for $n>1$ (i) and (ii) together imply

(iii) $f(Z)$ is bounded in H_n^+.

For $n=1$, however, (iii) should also be included in the definition of modular form.

From this definition, we find that the product nw must be an even integer. This follows since $M = E$ and $-M = -E$ both represent the identity transformation in Γ_n, yielding

$$f(-EZ) = f(Z)$$

and by (ii)

$$f(-EZ) = (-1)^{nw} f(Z).$$

Hence we have proved that modular forms not vanishing identically can exist only in the case $nw \equiv 0 \pmod 2$.

Additionally, it is known that modular forms of negative weight vanish identically and that modular forms of weight zero are necessarily constants. The first statement follows directly form the boundedness property of modular forms, while the latter requires more work and is found in [4].

B. Existence of Modular Forms (Eisenstein Series)

To prove the existence of modular forms, we will define a set of functions called Eisenstein series and show that they are indeed modular forms. We proceed by making the following definition for two matrices P and Q.

Definition 2: A pair of integral n-by-n matrices P and Q will be called coprime (denoted by [P,Q] if there exist integral matrices X and Y satisfying the relation; PX + QY = E. Further, matrices P and Q are said to form a symmetric pair if $PQ^T = QP^T$.

The Eisenstein series of degree n and weight w is then defined to be

$$\text{(I-2)} \qquad \phi_w(Z) = \sum_{[C,D]} |CZ+D|^{-w}$$

where the summation extends over all n-by-n coprime symmetric matrix pairs [C,D]. It can be proved that such a series converges absolutely and uniformly for all points $Z \in H_n^+$ provided that $w > n+1$ [5].

$\phi_w(Z)$ is a modular form of weight w. For apply successively the conditions of the definition:

(i) That $\phi_w(Z)$ is holomorphic at all points $Z \in H_n^+$ is assured by the uniform convergence of the series on compact subsets of H_n^+.

(ii) Let $M_1 = \begin{pmatrix} A_1 B_1 \\ C_1 D_1 \end{pmatrix} \in \Gamma_n$. Consider $\phi_w(Z_1)$, with

$$Z_1 = (A_1 Z + B_1)(C_1 Z + D_1)^{-1}$$

then

$$|CZ_1 + D| = |C(A_1 Z + B_1)(C_1 Z + D_1)^{-1} + D|$$

$$= |C_1 Z + D_1|^{-1} \; |C_2 Z + D_2|$$

where

$$C_2 = CA_1 + DC_1, \quad D_2 = CB_1 + DD_1.$$

Hence

$$\phi_w(Z_1) = |C_1 Z + D_1|^w \sum_{[C,D]} |C_2 Z + D_2|^{-w} \quad .$$

To prove that $\phi_w(Z_1) = |C_1 Z + D_1|^w \phi_w(Z)$, it suffices to prove that as $[C,D]$ runs over all coprime symmetric pairs, so does $[C_2,D_2]$.

First of all we have

$$(C_2D_2) = (CD)M_1, \quad (CD) = (C_2D_2)M_1^{-1}$$

implying that $[C_2,D_2]$ are coprime and distinct for different $[C,D]$. Moveover by equation (I-1),

$$M_1IM_1^T = I, \qquad M^TI^{-1}M = I^{-1}, \quad M^TIM = I,$$

$$CD^T - DC^T = (CD)I(CD)^T = (CD)M_1IM_1^T(CD)^T$$

$$= C_2D_2^T - D_2C_2^T,$$

hence $[C_2,D_2]$ is a symmetric pair if $[C,D]$ is. The result then follows.

(iii) The final condition for $\phi_w(Z)$ to be a modular form amounts to showing that the Eisenstein series (I-2) can be expressed as a Fourier series at ∞. The proof is a consequence of a generalized Lipschitz formula and can be found in [5].

Hence modular forms do exist, and we can give an explicit formula for them in the case of Eisenstein series. The importance of these Eisenstein series will become apparent as we develope Chapter II.

C. Fourier Expansion of Modular Forms

1. Definition 3: A matrix A is called properly (improperly) unimodular if and only if A and A^{-1} are both integral such that $|A| = +1 (|A| = -1)$ where $|\cdot|$ indicates determinant. Furthermore, a unimodular matrix is a matrix that is either properly unimodular or improperly unimodular.

For a matrix M_1 in Γ_n of the form $M_1 = \begin{pmatrix} U^T & S \\ 0 & U^{-1} \end{pmatrix}$ where S is an integral symmetric matrix and U is a unimodular matrix, we can infer from Definition 1 that $f(Z+S) = f(Z)$ for $f \in (\Gamma_n, w)$ and $U = E$. In particular, this implies that modular forms are periodic with period one in every element $z_{k\ell}$ of $Z \in H_n^+$. Hence, we can develop the Fourier series of a modular form,

$$\text{(I-3)} \qquad f(Z) = \sum_T a_w(T) e^{2\pi i \sigma(TZ)}$$

where the summation extends over all semi-integral matrices T (i.e., the off-diagonal elements are half-integers) and $\sigma(\cdot) = \text{trace } (\cdot)$.

Letting $S = 0$ in the above matrix M_1 and again using Definition 1, we see that for $f \in (\Gamma_n, w)$,

$$f(Z) = f(Z[U]) \tag{I-4}$$

where $Z[U] = U^T Z U$. Also by observing that the trace of a matrix product is invariant under a cyclic permutation in the succession of the factors (i.e., $\sigma(ABC) = \sigma(CAB)$), we have

$$\begin{aligned} f(Z[U]) &= \sum_T a_w(T) e^{2\pi i \sigma(TU^T ZU)} \\ &= \sum_T a_w(T) e^{2\pi i \sigma(UTU^T Z)} \\ &= \sum_T a_w(T[(U^T)^{-1}]) e^{2\pi i \sigma(TZ)} \end{aligned} \tag{I-5}$$

Therefore a comparison of the coefficients of (I-3) and (I-5) by means of (I-4) allows us to conclude by the uniqueness of Fourier expansions that

$$a_w(T) = a_w(T[(U^T)^{-1}])$$

where U is an arbitrary unimodular matrix. Replacing $(U^T)^{-1}$ by U, we get

$$a_w(T) = a_w(T[U]). \tag{I-6}$$

We exploit this relationship in the following definition.

Definition 4: Two matrices A and B are said to be unimodularly equivalent if for some unimodular matrix U, we have $B = A[U]$ where $A[U] = U^T AU$. In this case B is said to be a unimodular transformation of A.

This defines an equivalence relation on the set of semi-integral matrices T. Relation (I-6) then implies that the coefficient $a_w(T)$ depends only upon the unimodular equivalence class (T) to which T belongs. Here, of course, (T) denotes the equivalence class for the matrix T.

Separating the semi-integral matrices into classes (T) and letting

$$f_{(T)}(Z) = \sum_{T_1 \in (T)} e(T_1 Z)$$

where

$$e(T_1 Z) = e^{2\pi i \sigma (T_1 Z)},$$ equation (I-3) becomes

$$f(Z) = \sum_{(T)} a_w(T) f_{(T)}(Z)$$

summed over all the different classes of semi-integral matrices.

It is known from the boundedness property of modular forms that $a_w(T) = 0$, unless $T \geq 0$ (semi-positive definite) [4]. As a consequence of this, we write (I-7) as

$$\text{(I-8)} \qquad f(Z) = \sum_{(T)\geq 0} a_w(T) f_{(T)}(Z)$$

We also define $\mathfrak{T}_n$ to be the set of all n-by-n semi-integral, semi-positive definite matrices, and $\mathfrak{T} = \mathfrak{T}_2$.

2. For the applications we have in mind (namely differential equations involving the Fourier Series of modular forms - Part Two), we will need the expression for the Fourier series of a modular form raised to the k-th power. The result for $f \in (\Gamma_n, w)$ expressed as in equation (I-8) is

$$\text{(I-9)} \qquad \begin{aligned} f^k(Z) &= \sum_{(D)\geq 0} \prod_1^k a_w(T_i) C_D f_{(D)}(Z) \\ D &= \sum_1^k T_i \end{aligned}$$

where C_D = number of solutions of $D = \sum_1^k T_i$ and $T_i \in \mathfrak{T}_n$.

This follows directly by noting that

$$f_T(Z)\ f_S(Z) = \sum_{\substack{T_1+S_1 \\ T_1\in(T),\ S_1\in(S)}} e[(T_1+S_1)Z]. \tag{I-10}$$

Since the sum of two matrices in $\mathfrak{J}_n$ is again in $\mathfrak{J}_n$, the equation

$$T_1+S_1 = D_1, \quad T_1 \text{ and } T_2\in\mathfrak{J}_n$$

holds for only a finite number of semi-integral matrices T_1 and S_1. Suppose it has in all C_{D_1} solutions. Then, since

$$U^T T_1 U + U^T S_1 U = U^T D_1 U, \qquad \text{U unimodular},$$

the equation has the same number of solutions for all matrices D_1 unimodularly equivalent to D. Hence every element of the class D appears in the sum (I-10) the same number of times and equation (I-9) follows by induction on k.

D. Siegel Operator

In this section we will define an operator, Φ, which maps the modular forms of degree $n>1$

to those of degree n-1 with the

same weight: i.e., $\Phi:(\Gamma_n,w)\to(\Gamma_{n-1},w)$. From the action of this operator, we will be able to determine a relation between some Fourier coefficients of degree n and those of degree n-1.

Toward this end, it is a straightforward verification to show that if $Z = \begin{pmatrix} Z_1 & * \\ * & * \end{pmatrix} \in H_n^+$, then the matrix Z_1 obtained by cancelling the last row and column of Z is an element of H_{n-1}^+. Conversely, for every $Z_1 \in H_{n-1}^+$, we have $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & i\lambda \end{pmatrix} \in H_n^+$ provided $\lambda > 0$. Then for $f \in (\Gamma_n, w)$, we have the following proposition defining the <u>Siegel</u> <u>Operator</u> Φ[4].

<u>Proposition</u> <u>1</u>: Let $f \in (\Gamma_n, w)$; then $\lim_{\lambda\to\infty} f\begin{pmatrix} Z_1 & 0 \\ 0 & i\lambda \end{pmatrix}$ exists and is a modular form of degree n-1 and weight w. This limit function is denoted by $\Phi(f(Z))$.

Implicit in the proof of the proposition is that the terms of $f(Z) = \sum_{(T)\geq 0} a_w(T) f_{(T)}(Z) \in (\Gamma_n, w)$ involving matrices T for which $|T| \neq 0$ vanish in the limit and only those terms for which $|T| = 0$ survive. We then obtain

$$\Phi(f(Z)) = \sum_{(T_1)\geq 0} a_w(T_1) f_{(T_1)}(Z) \quad \in (\Gamma_{n-1}, w)$$

where T_1 is an n-1 by n-1 semi-integral matrix and where by definition

$$\text{(I-11)} \qquad a_w(T_1) = a_w \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In the case that we restrict ourselves to Eisenstein series of degree 2, we can obtain an explicit expression for the Fourier coefficients associated with matrices of determinant zero. Indeed, if we let T be a semi-integral matrix of order 2 such that $|T| = 0$, it is not hard to see that there exists a unimodular matrix U such that

$$T[U] = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} : m \text{ integral, } > 0.$$

(An explicit method for determining the U is given in section F of this chapter.)

Now if for this T, $a_w(T)$ is a Fourier coefficient of an Eisenstein series of degree 2, then by equation (I-11)

$$a_w(T) = a_w(m)$$

where $a_w(m) =$ the Fourier coefficient for the Eisenstein series of the classical modular group, $n=1$. In this case, the coefficients of $\phi_w \in (\Gamma_1, w)$ are well known to be

(I-12) $$a_w(m) = \frac{(-1)^{w/2} 2w}{B_w} \sigma_{w-1}(m)$$

where

$$\sigma_k(m) = \sum_{\substack{d|m \\ d>0}} d^k \text{ (the elementary divisor function),}$$

and B_n are the Bernoulli numbers, defined for instance by

(I-13) $$Z \cot Z = 1 - \sum_{n=1}^{\infty} \frac{B_n 2^{2n} Z^{2n}}{(2n)!} .$$

For reference we note that $\frac{(-1)^{w/2} 2w}{B_w} = 240, -504, -264$, and $54,600/691$ for $w = 4, 6, 10$, and 12 respectively.

In concluding this present section, we distinguish those modular forms $f \in (\Gamma_n, w)$ which have no terms in T for which $|T| = 0$.

Definition 5: A function $f \in (\Gamma_n, w)$ is called a cusp form if

$$\Phi(f(Z)) = 0$$

i.e., f is in the kernel of the Siegel Φ operator.

E. Witt's Identity

E. Witt [12] proved a theorem in 1939 from which we will obtain an identity between a sum of degree two Fourier coefficients and a product of degree one coefficients. The importance of this identity (to our paper) is that it bears some geometric similarity to the second conjecture in Chapter II. Further, it provides a check which we use when calculating degree two coefficients.

If f_1 and $f_2 \in (\Gamma_n, w)$, then $f = af_1 + bf_2$ where a,b are any complex constants is also a modular form of the same weight. Therefore, the collection of modular forms, (Γ_n, w), of fixed weight form a vector space over the complex numbers. It is well known that this space is finite dimensional. In particular, if we let $\rho(w)$ denote the number of linearly independent modular forms in (Γ_1, w), we have

$$\text{(I-14)} \qquad \rho(w) = \begin{cases} [\frac{w}{12}] & , w \equiv 2 \bmod 12 \\ [\frac{w}{12}]+1, & w \not\equiv 2 \bmod 12 \end{cases}$$

where $[x]$ denotes the greatest integer $\leq x$.

Witt's theorem can be stated as follows:

Proposition 2 (Witt): For every modular form $f \in (\Gamma_n, w)$, we have the expansion

$$f\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} f_i(Z_1) g_j(Z_2)$$

where the $f_i (1 \leq i \leq m)$ and $g_j (1 \leq j \leq n)$ are linearly independent modular forms in (Γ_{n_1}, w) and (Γ_{n_2}, w) respectively and $n_1 + n_2 = n$.

For $f \in (\Gamma_2, w)$ and $w = 2, 4, 6, 8, 10$, equation (I-14) yields $\rho(w) = 1$. Hence the theorem implies the following expansion

$$f\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} = f_1(Z_1)\ f_1(Z_2)$$

where

$$f_1 \in (\Gamma_1, w).$$

Expressing f and f_1 by their Fourier series yields, by uniqueness of Fourier expansions, the <u>Witt identity</u>:

$$a_w(t_1)a_w(t_2) = \sum_{t_3} a_w \begin{pmatrix} t_1 & t_3/2 \\ t_3/2 & t_2 \end{pmatrix} \qquad \text{(I-15)}$$

for fixed t_1, t_2. The sum is taken over all $|t_3| \leq 2\sqrt{t_1 t_2}$ and $a_w(t_i)$, $i = 1, 2$, are the Fourier coefficients for the classical modular form f_1.

F. <u>Minkowski Reduced Domains</u>

In concluding this present chapter of preliminary remarks, we will look a bit more closely at the set $\mathfrak{T}$ of all semi-integral matrices of order 2 such that $T \geq 0$. It was shown in section B that to completely specify the Fourier series of a modular form, we need only determine the coefficients associated with classes of unimodularly equivalent matrices in $\mathfrak{T}$. By appealing to the classical quadratic reduction theory of Minkowski and Gauss, we will be able to fix in each class a typical representative satisfying certain extremal properties which will be useful for our purposes.

Consider all real symmetric matrices $A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$.

With each matrix A associate a point in 3-dimensional real vector space with coordinates (a,c,b). For simplicity of notation, we shall use this triplet (a,c,b) to denote the matrix A. We also note that the set of points corresponding to the matrices $T \in \mathfrak{T}$ are a lattice in this real space.

Definition 6: A Minkowski reduced matrix is a matrix $T = (a,c,b) \in \mathfrak{T}$ satisfying the condition

$$|b| \leq a < c \quad \text{or, if} \quad a = c \text{ then } 0 < b \leq a \; .$$

The set of all Minkowski reduced matrices will be denoted by R_M. We have the following lemma concerning R_M [11].

Lemma 1: No two Minkowski reduced matrices are properly unimodularly equivalent.

The next proposition tells us that for every $T \in \mathfrak{T}$ there exists a unique properly unimodular equivalent matrix in R_M.

Proposition 3 (Minkowski): Let $T = (a,c,b)$ be an arbitrary element of $\mathfrak{T}$ with $\tilde{\Delta} = 4ac - b^2 \geq 0$, then there exists a finite succession of proper unimodular transformations which will reduce T to a unique element of R_M (called the reduced matrix for T).

Proof: Clearly if T is already reduced, the unimodular matrix E will work. Therefore, let $T \notin R_M$, $S_i = \begin{pmatrix} 1 & n_i \\ 0 & 1 \end{pmatrix}$, and $W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For some integer n_1,

$$T_1 = T[S_1] = (a,\ n_1^2 a + n_1 b + c,\ 2n_1 a + b)$$

$$= (a_1,\ c_1,\ b_1)$$

can be determined such that $-a_1 < b_1 \leq a_1$.

Case I: $a_1 < c_1$ implies we are finished, since we have chosen b_1 such that $-a_1 < b_1 \leq a_1$.

Case II: $a_1 = c_1$, but $-a_1 \leq b_1 < 0$, then $T_2 = T_1[W]$ is reduced since the application of W takes $(a_1, c_1, b_1) \rightarrow (c_1, a_1, -b_1) = (a_2, c_2, b_2)$ implying $T_2 \in R_M$.

Case III: $c_1 < a_1$, then if $T_2 = T_1[W]$, we have $c_2 < a_2$ by the action of W. If $-a_2 < b_2 \leq a$, we're through. If not, there exist an $n_2 \in \mathbb{Z}$ such that $T_3 = T_2[S_2]$ has $-a_3 < b_3 \leq a_3$. If T_3 is not reduced, apply W again and get $T_4 = T_3[W]$; thus having repeated the entire procedure over again. What remains to be shown is that for some k, T_k will be reduced, i.e., after finitely many steps, this process leads to a matrix $T_k \in R_M$.

To this end, note that $a_k^2 \geq \tilde{\Delta}_k = 4\ a_k c_k - b_k^2$ implies $a_k^2 - 4a_k c_k + b_k^2 \geq 0$ or $a_k \geq 2c_k = 2a_{k+1}$ under the action of W where $-a_k < b_k \leq a_k$. Hence as long as $\tilde{\Delta}_1 \leq a_k^2$, $a_k \geq 2a_{k+1}$ so that we will always be considering Case III at each repetition of the procedure. Further, starting with some positive number for a, a finite number of applications of S followed by W will produce a number $a_k^2 < \tilde{\Delta}$.

So suppose then that we are at the point such that

$$-a_k < b_k \leq a_k, \quad \tilde{\Delta} > a_k^2, \quad c_k < a_k.$$

Then on application of W, we have

$$T_{k+1} = T_k[W] = (a_{k+1}, c_{k+1}, b_{k+1}) = (c_k, a_k, -b_k)$$

and on application of S,

$$T_{k+2} = T_{k+1}[S] = (a_{k+2}, c_{k+2}, b_{k+2}) = (c_k, n^2 c_k - b_k n + a_k, 2nc_k - b_k)$$

where n is determined such that $-a_{k+2}<b_{k+2}\le a_{k+2}$. Now if $|n|\ge 2$, we have $c_{k+2}\ge a_{k+2}$ since

$$c_{k+2} = \frac{4a_k^2(n^2c_k-nb_k+a_k)+(n^2b_k^2+n^2b_k^2)}{4a_k^2}$$

$$= a_k\left[\left(\frac{nb_k}{2a_k}-1\right)^2+\frac{n^2(4a_kc_k-b_k^2)}{4a_k^2}\right]$$

$$\ge a_k[0+4\cdot\tfrac{1}{4}] = a_k\ge c_k = a_{k+2} .$$

If $|n| = 0$, we have $c_{k+2}>a_{k+2}$, since

$$c_{k+2} = a_k>c_k = a_{k+2} .$$

Now $|n| = 1$, we cannot have $c_{k+2}<a_{k+2}$, since then we would have

$$c_{k+2} = c_k-b_k+a_k<c_k = a_{k+2}$$

implying that $a_k<b_k$ contrary to the assumption on a_k and b_k. Hence in all cases, $c_{k+2}\ge a_{k+2}$ and

$-a_{k+2}<b_{k+2}\leq a_{k+2}$. If T_{k+2} is still not in R_M (which may happen if $a_{k+2} = c_{k+2}$), then $T_{k+3} = T_{k+2}[W]\in R_M$.

Uniqueness of the reduced matrix for T follows from Lemma 1.

QED

An immediate corollary is

Corollary 1: Let T be an arbitrary semi-positive definite matrix and let T_R be the Minkowski reduced matrix for T. Then

$$\sigma(T)\geq\sigma(T_R)$$

equality holding if and only if $T = T_R$.

The proposition along with the lemma assures us that R_M contains only one representative from each properly unimodular equivalence class (T). In addition, in the proof of the proposition we are given an explicit method for reducing a matrix $T\in\mathfrak{J}$ to an element of R_M (and hence to its associated equivalence class).

Recalling that coefficients of modular forms are invariant under unimodular equivalences (I-6), we find that the set R_M is too large for our purposes.

We are able to obtain a subset R of R_M which will be sufficient for our needs. Here, we get

$$R = \{ T = (a,c,b): T \in \mathfrak{T} \text{ and } 0 \le b \le a \le c \} \quad .$$

This comes by identifying elements of R_M by the improper unimodular transformation $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Its action identifies matrices (a,c,b) with matrices (a,c,-b).

We see that R contains one representative from each class of unimodularly equivalent positive definite matrices which is Minkowski reduced and has positive off-diagonal element. We shall call R the set of unimodularly reduced matrices in $\mathfrak{T}$.

2. A useful geometric interpretation of the semi-positive, semi-integral matrices $\mathfrak{T}$ is given in the following [11].

<u>Proposition 4</u>: For 2-by-2 symmetric real matrices A, let $S = \{A = A \ge 0\}$ containing the lattice $\mathfrak{T}$. Then we have

(a) The space S is a convex circular cone with vertex at the origin in Euclidean three space,

(b) The lattice R_M is contained in a convex pyramid with vertex at the origin.

Noting that for matrices $T = (a,c,b) \geq 0$, a and c must be positive; we have pictorially,

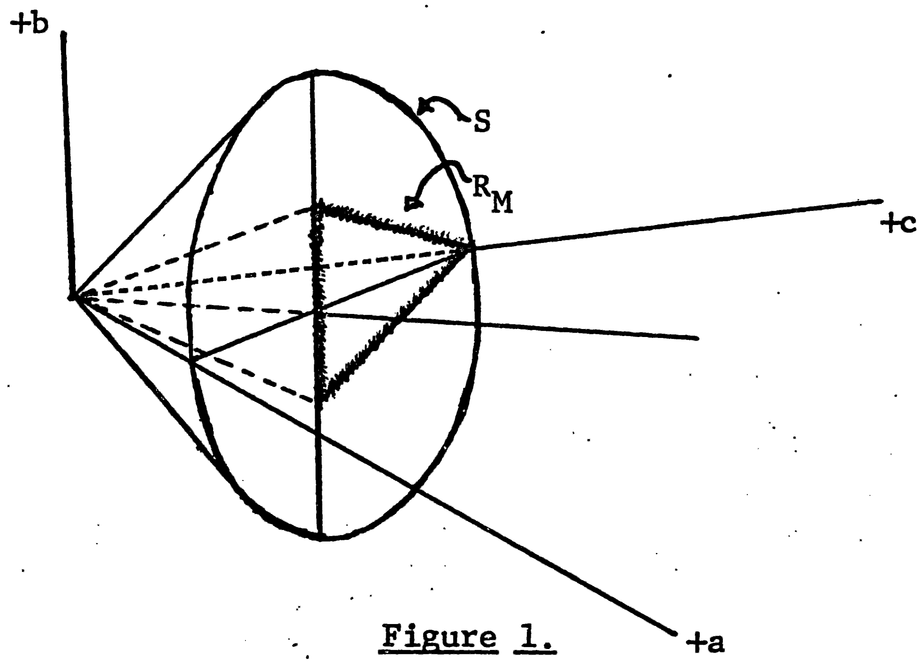


Figure 1.

Definition 7: For a fixed number α, define the α-trace plane to be $P_\alpha = \{A:\ A$ is real symmetric and $\sigma(A) = \alpha\}$ where $\sigma(A)$ denotes the trace of A.

With respect to Figure 1, we see that the set of all semi-positive definite matrices $T \in P_\alpha$ lie within a closed disk of radius α, for if $T = (a,c,b) \geq 0$, then

$$ac - b^2/4 \geq 0$$

so that

$$4ac + a^2 + c^2 \geq b^2 + a^2 + c^2,$$

or

$$(c^2 - 2ac + a^2) + b^2 \leq a^2 + 2ac + c^2,$$

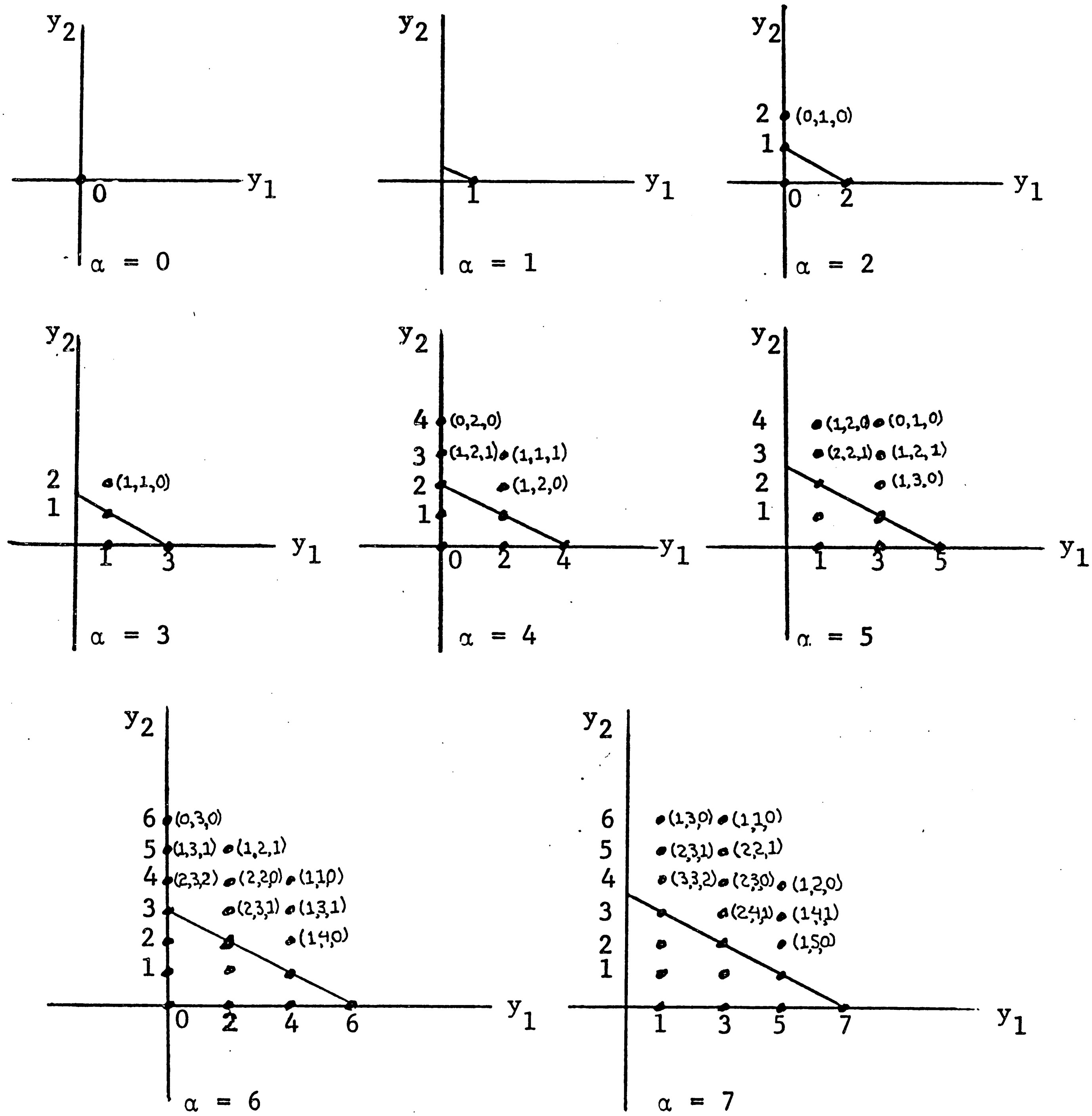
and so

$$(c-a)^2 + b^2 \leq (a+c)^2 = \sigma(T) = \alpha.$$

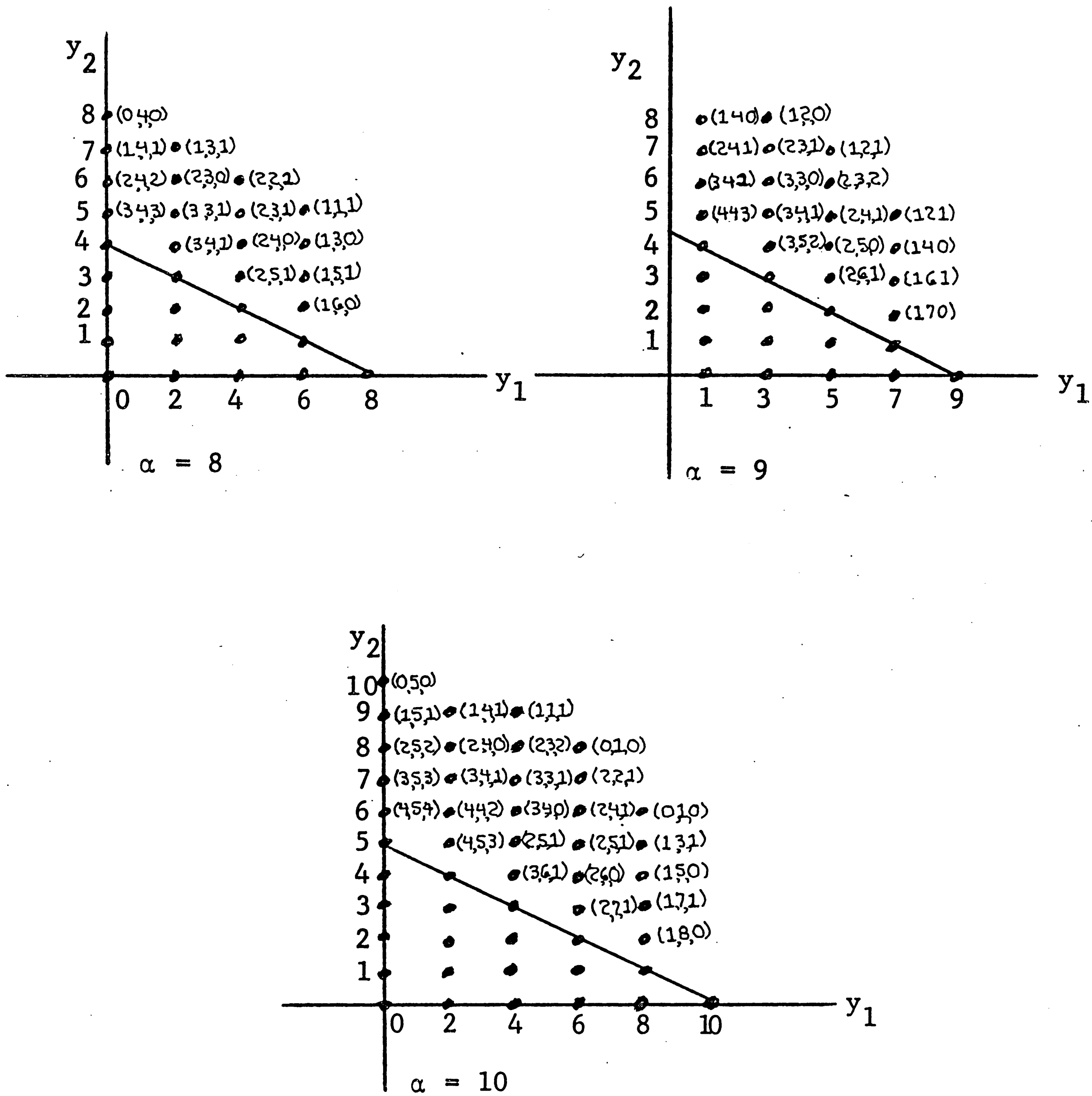
The first quadrant for a few of these trace planes with coordinates $y_1 = c-a$ and $y_2 = b$ is given in figure 2. The lattice points in the figure represent elements of $P_\alpha \cap \mathfrak{T}$ and the points within the triangles, elements of $P_\alpha \cap R$. Points outside the triangles have been identified by the reduced matrices to which they are equivalent.

The results of the previous section tell us that we need only evaluate the Fourier coefficients of modular forms on the lattice points in the first quadrant of each trace plane; and in particular on lattice points within the closed reduced triangle in quadrant 1.

Figure 2: Lattice Points by Trace Planes - P_α



(Figure 2: Cont.)



II. The Two Conjectures for Fourier Coefficients

In this chapter and for the remainder of the paper, we restrict ourselves to H_2^+ and the Siegel modular group $\Gamma = \Gamma_2$. It is the purpose of this chapter to present and derive consequences of two conjectures concerning the Fourier coefficients $a_w(T)$ for $f \in (\Gamma_2, w)$. We begin by formalizing the definition of primitive and imprimitive matrices. Let $\mathfrak{T}$ be the set of all semi-integral, semi-positive matrices.

Definition 8: For $T = (a,c,b) \in \mathfrak{T}$, we call $d \in \mathbb{Z}^+$ a divisor of T if the matrix T/d defined by $T/d = (a/d, c/d, b/d)$ is an element of $\mathfrak{T}$.

Definition 9: A matrix $T \in \mathfrak{T}$ is called a primitive matrix if the only divisor of T is 1; otherwise T is called an imprimitive matrix.

A. Conjectured Equation for Imprimitive Matrices:

Let $T = (a,c,b) \in \mathfrak{T}$ with $|T| = \Delta = ac - \frac{b^2}{4}$. We know by Appendix I (Proposition A1) that primitive matrices with the same determinant have equal Fourier coefficients in the expansion of an Eisenstein series. With this in mind, we form $\mathfrak{M}_\Delta \subset \mathfrak{T}$ where $\mathfrak{M}_\Delta = \{$all primitive matrices in $\mathfrak{T}$ with determinant$= \Delta\}$.

Also since $4\Delta \equiv 0 \pmod 4$ for $\Delta \in \mathbb{Z}^+$ and $4\Delta \equiv 3 \pmod 4$ for $\Delta \notin \mathbb{Z}^+$, we choose the matrices $(1,\Delta,0)$ and $(1,\frac{4\Delta+1}{4},1)$ as representatives for $\mathfrak{N}_\Delta$, dependent on whether $\Delta \in \mathbb{Z}^+$ or $\Delta \notin \mathbb{Z}^+$, respectively.

For an arbitrary matrix $T \in \mathfrak{T}$ with determinant Δ, define

$$\text{(II-1)} \quad [T]_p = \begin{cases} (1,\Delta,0) & ,\Delta \in \mathbb{Z}^+ \\ (1,\frac{4\Delta+1}{4},1), & \Delta \notin \mathbb{Z}^+ \end{cases}$$

Now with these preliminaries, we can state the first conjecture of this paper.

Conjecture I (Imprimitive matrix): Let $T \in \mathfrak{T}$. Then for $f \in (\Gamma_2, w)$ with Fourier coefficients $a_w(T)$, we have

$$\text{(II-2)} \qquad a_w(T) = \sum_{d|T} d^{w-1} a_w([\tfrac{T}{d}]_p)$$

where d is a divisor of T.

Remarks

(1) First of all, we should like to emphasize that the conjecture as stated is an expression for all

forms $f\in(\Gamma_2,w)$ where w is arbitrary. In particular, if application of (II-2) is made to the coefficients of Table I (Part Two - Chapter V), we find the conjecture holds. This table lists Fourier coefficients for the two Eisenstein series ϕ_4 and ϕ_6 and the two cusp forms χ_{10} and χ_{12}.

(2) In the case that $a_w(T)$ is the Fourier coefficient for an Eisenstein series in (Γ_2,w) and $T = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}\in\mathfrak{J}$, (II-2) yields the elementary divisor solution as it should (Chapter I-Section D).

(3) No information on the values of the Fourier coefficients of primitive matrices is gotten from (II-2). Consequently, to evaluate (II-2) one must know the value of the coefficients on the primitive representatives $(1,\Delta,0)$ and $(1,\frac{4\Delta+1}{4},1)$.

(4) As pointed out in the introduction to this paper, both Siegel [10] and Maass [6] have obtained equations for the Fourier coefficients of Eisenstein series associated with imprimitive matrices (Appendix I- equations (AI-1) and (AI-3), respectively). A glance at these equations gives the obvious conclusion that they are more complex than (II-2). Moreover, they hold only for Eisenstein series.

(5) There is a close similarity in form between equation (II-2) and equation (AI-3) of Appendix I. This latter equation, which was derived from the Hecke operator theory [6], does not appear to be a consequence of (II-2) nor vice versa. Nevertheless, this strongly suggests that a proof of this conjecture might lie within the Hecke operator theory of Maass or an extention of this theory.

(6) Let $T = (t_1,t_2,t_3) \in \mathfrak{T}$ and define $e = e(T) = \text{g.c.d}(t_1,t_2,t_3)$ and $D = D(T) = |2T|/e^2$. Then if Conjecture I is true, Maass' Conjecture (Appendix I) is true. That is, $a_w(T)$ is a function only of the parameters $e(T)$ and $D(T)$; so that we can write $a_w(T) = \alpha_w(e,D)$. Moreover, the conjecture yields the following explicit expression for the function $\alpha_w(\cdot,\cdot)$ in terms of $\alpha_w(\cdot,\cdot)$ evaluated on the primitive matrices:

$$\alpha_w(e,D) = \sum_{d|e} d^{w-1}\alpha_w(1,\frac{e^2D}{d^2}).$$

This is a straight forward application of the definitions of e and D and the fact that coefficients of primitive matrices with the same determinant are equal.

B. <u>A Witt Type Conjecture for the ϕ_4 Coefficients</u>:

In this section we will state a conjectured identity between the Fourier coefficients of the Eisenstein series ϕ_4 of degree 2 and ϕ_4 of degree 1. It will be seen that the conjecture bears a marked geometric similarity to Witt's identity (I-15). Moreover, in the next section we will show that this conjecture yields a recursive equation for the ϕ_4-coefficients (of degree 2) associated with primitive matrices $T \in \mathfrak{J}$. This is just what is needed to completely determine the ϕ_4-coefficients in the degree 2 case by Conjecture I.

Without confusion, we will let $a_w(\cdot)$ be the Fourier coefficient for ϕ_4 of degrees one and two. In the former case, the argument of $a_w(\cdot)$ will be an integer, and in the latter, an element of $\mathfrak{J}$. Also for fixed integers α and b, we define

$$T_{\alpha,b} = \{T = (\tilde{a},\tilde{c},\tilde{b}) \in \mathfrak{J} : \sigma(T) = \alpha \text{ and } \tilde{b} = b\}.$$

<u>Conjecture II</u> (<u>ϕ_4-identity</u>): If α and b are fixed integers satisfying $\alpha \geq |b| \geq 0$, then

$$\left.\begin{array}{ll} \text{(II-3)} & a_4(\alpha-b)a_4(\alpha+b) \\[1em] \text{(II-4)} & a_4(\alpha-b)[a_4(\alpha+b)-a_4(\frac{\alpha+b}{2})] \\[1em] \text{(II-5)} & a_4(\alpha+b)[a_4(\alpha+b)-a_4(\frac{\alpha-b}{2})] \end{array}\right\} = 120 \sum_{T\in\mathfrak{J}_{\alpha,b}} a_4(T)$$

$$\begin{cases} \text{if } (\alpha-b)\equiv 1,3 \pmod 4 \\ \text{if } (\alpha-b)\equiv 2 \pmod 4 \\ \text{if } (\alpha-b)\equiv 0 \pmod 4\text{, } b \text{ odd} \end{cases}$$

where $a_4(.)$ are the Fourier coefficients for ϕ_4.

<u>Remarks</u>:

(1) Unlike the first conjecture, this one is a statement about the ϕ_4-Fourier coefficients only. For the coefficients of Tables I and II (Part Two-Chapter V), the conjecture is found to hold.

(2) No immediate generalizations of these equations to forms of other weights (if they exist) are known at this time. Perhaps, since ϕ_4 is distinguished as the generator of the graded ring of degree 2 modular forms [7] (and Part Two of this paper), we cannot expect such a simple identity for other forms.

(3) Equations (II-3) and (II-4) hold for b both even and odd, whereas (II-5) holds only for b odd. The equation for $(\alpha-b)\equiv 0 \pmod 4$ and b even has eluded us and consequently none is known in this case. Nevertheless, as we shall see in the next section, equation (II-3) alone is sufficient to generate all the ϕ_4-coefficients for primitive matrices, and thus all ϕ_4-coefficients.

(4) As mentioned at the head of this section, the conjecture bears a geometric similarity to Witt's identity. In order to see this, we write Witt's identity (I-15) for ϕ_4-coefficients as

$$a_4(a)a_4(c) = \sum_{|b|\leq 2\sqrt{ac}} a_4(a,c,b), \text{ for fixed } a,c.$$

We note that fixing a and c in this equation amounts to fixing the trace plane P_α with $\alpha = a+c$. In this context the sum on the right is a sum of the coefficients associated with all lattice points on a line segment parallel to the y_2 axis in the trace plane coordinates

$$y_1 = c-a \quad \text{and} \quad y_2 = b \quad \text{of figure 2.}$$

In contrast to this, equations (II-3,4,5) are statements about the sum of coefficients associated with all the lattice points on a line segment, $T_{\alpha,b}$, parallel to the y_1 axis. Hence the geometric similarity between Witt's identity and Conjecture II lies in the geometry of the lattice points taken in their respective sums. In addition, both of these sums are equal to a sum of products of classical ϕ_4-coefficients.

Since the equations of the conjecture do not appear to be direct consequences of Witt's Theorem, the comparison statements above are intended only to show that Conjecture II has a certain "naturality" within the framework of known results.

C. A Theorem for the Primitive Matrix Coefficients of ϕ_4:

If the first conjecture were true (II-2), then all that remains to completely determine the Fourier coefficients of a modular form is to determine the coefficients for the primitive representatives $(1,\Delta,0)$ and $(1,\frac{4\Delta+1}{4},1)$. In the case that the modular form is the Eisenstein series of weight 4, ϕ_4, we will show that some rather simple recursive equations do exist for these primitive ϕ_4-coefficients if Conjecture II is valid.

1. Toward this end, we prove a general lemma.

Lemma 2: Let $a_w(\cdot)$ be the Fourier coefficients for $f \in (\Gamma_2, w)$. If α and b are fixed integers satisfying $\alpha \geq |b| \geq 0$ with $b \equiv m \pmod 2$ and $\alpha - b \equiv n \pmod 4$, then there exists integers $\tilde{a} \geq 0$ and $\tilde{c} \geq -\left(\frac{2m+n}{4}\right)$ such that

$$\sum_{T \in \mathfrak{J}_{\alpha,b}} a_w(T) = \begin{cases} \displaystyle\sum_{\frac{\tilde{a}(4\tilde{c}+(2m+n))}{4} \geq i^2} a_w\left(\tilde{a}, \tilde{c} - i + \frac{\tilde{a}+(2m+n)}{4}, \tilde{a} - 2i\right) \\ \displaystyle\sum_{\frac{\tilde{a}(4\tilde{c}+(2m+n))-1}{4} \geq i^2+i} a_w\left(\tilde{a}, \tilde{c} - i + \frac{\tilde{a}+(2m+n)-2}{4}, \tilde{a} - 2i - 1\right) \end{cases}$$

$$\text{for} \quad \begin{cases} m+n \equiv 0 \pmod 2 \\ m+n \equiv 1 \pmod 2 \end{cases}.$$

Proof: Let α and b be fixed integers satisfying the hypothesis of the lemma. For every $T = (a,c,b) \in \mathfrak{J}_{\alpha,b}$ we have $\sigma(T) = a+c = \alpha$; so that $a = \frac{\alpha-(c-a)}{2}$ and $c = \frac{\alpha+(c-a)}{2}$. Using this, we get

$$\sum_{T \in \mathfrak{J}_{\alpha,b}} a_w(T) = \begin{cases} \displaystyle\sum_{\frac{\alpha^2-b^2}{4} \geq i^2} a_w\left(\frac{\alpha}{2} - i, \frac{\alpha}{2} + i, b\right) \\ \displaystyle\sum_{\frac{\alpha^2-b^2+1}{4} \geq i^2+i} a_w\left(\frac{\alpha-1}{2} - i, \frac{\alpha+1}{2} + i, b\right) \end{cases}$$

$$\text{for} \quad \begin{cases} (c-a) \equiv 0 \pmod 2 \\ (c-a) \equiv 1 \pmod 2. \end{cases}$$

Now since the unimodular transformation $U = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ takes matrices (a,c,b) to matrices $(a+c-b,\ a,\ 2a-b)$, and since the Fourier coefficients of modular forms are invariant under unimodular transformations, we rewrite the above equation as

(II-6)

$$\sum_{T \in \mathfrak{J}_{\alpha,b}} a_w(T) = \begin{cases} \displaystyle\sum_{\frac{\alpha^2-b^2}{4} \geq i^2} a_w(\alpha-b, \tfrac{\alpha}{2}-i, (\alpha-b)-2i) \\[2ex] \displaystyle\sum_{\frac{\alpha^2-b^2+1}{4} \geq i^2+i} a_w(\alpha-b, \tfrac{\alpha-1}{2}-i, (\alpha-b)-2i-1) \end{cases}$$

$$\text{for} \quad \begin{cases} (c-a) \equiv 0 \pmod 2 \\ (c-a) \equiv 1 \pmod 2. \end{cases}$$

Next we note that for $\alpha-b\equiv n \pmod 4$ and $b\equiv m \pmod 2$ we have $c-a\equiv(n+m) \pmod 2$ and $\alpha+b\equiv(2m+n) \pmod 4$. This latter congruence implies that there exists an integer $\tilde{c} = \frac{\alpha+b-(2m+n)}{4}$. Coupling this with the fact that $\alpha\geq|b|\geq 0$ implies $\tilde{c}\geq-(\frac{2m+n}{4})$. Finally, we introduce the other parameter $\tilde{a}$ of the lemma by defining $\tilde{a}=\alpha-b\geq 0$. This completes the proof, since substitution of the parameters $\tilde{a}$ and $\tilde{c}$ in equation (II-6) yields the desired result.

QED

With this lemma as preparation, we prove the main result of this section. We recall that $a_w(T) = a_w(m)$ if $T = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$ (section D, Chapter I). Note that this is in agreement with the convention that $a_w(\cdot)$ be used for both degree 1 and 2 coefficients.

<u>Theorem 1</u> (<u>Primitive Matrix</u>): Let c be an arbitrary integer. Then if Conjecture II is true, we have

$$\text{(II-7)} \qquad a_4(1,\Delta,0) = a_4(4\Delta+1) - \sum_{\substack{i^2+i\le\Delta \\ 0<i}} a_4(1,\Delta-i^2-i,0)$$

where $\Delta = c\ge 0$, and

$$\text{(II-8)} \quad a_4(1,\tfrac{4\Delta+1}{4},1) = 2[a_4(4\Delta)- \sum_{i^2<\frac{4\Delta+1}{4}} a_4(1,\tfrac{4\Delta+1}{4}-i^2,1)]$$

where $\Delta = \frac{4c-1}{4}$ and $c\ge 1$.

<u>Proof</u>: Let $Q = \{(\alpha,b):\alpha,b\in\mathbb{Z},\alpha\geq b\ge 0$ and $\alpha-b = 1\}$.

Now as the tuplets (α, b) range over Q, Lemma 2 and equation (II-3) of Conjecture II imply

$$\text{(II-9)} \qquad a_4(1)a_4(4c+2m+1) = 120 \begin{cases} \displaystyle\sum_{c+\frac{2m+1}{4} \geq i^2} a_4(1, c-i+\tfrac{2m+2}{4}, 1-2i) \\ \displaystyle\sum_{c+\frac{m}{2} \geq i^2+i} a_4(1, c-i+\tfrac{m}{2}, -2i) \end{cases}$$

$$\text{for} \qquad \begin{cases} m \equiv 1 \pmod 2 \\ m \equiv 0 \pmod 2 \end{cases}$$

holds for every integer $c \geq -(\frac{2m+1}{4})$.

From (I-12) we have $a_4(1) = 240$, and thus from (II-9) we obtain for $m = 0$ and for every $c \geq 0$,

$$2a_4(4c+1) = \sum_{c \geq i^2+i} a_4(1, c-i, -2i),$$

and for $m = -1$ and for every $c \geq 1$,

$$2a_4(4c-1) = \sum_{c > i^2} a_4(1, c-i, 1-2i).$$

Next consider the unimodular transformation

$S_n = \begin{pmatrix} 1 & -n \\ 0 & -1 \end{pmatrix}$ which takes a matrix $A = (a,c,b)$ into

$$A[S_n] = (a, n^2a+nb+c, -2na-b).$$

In particular, for the matrix S_n with $n = i$, we have

$$(1,c-i,-2i)\ [S_i] = (1,c-i^2-i,0)$$

and

$$(1,c-i,1-2i)\ [S_i] = (1,c-i^2,1).$$

Since Fourier coefficients of modular forms are invariant under unimodular transformations, we have under the action of S_i:

$$2a_4(4c+1) = \sum_{c \geq i^2+i} a_4(1,c-i^2-i,0)$$

for every $c \geq 0$, and

$$2a_4(4c-1) = \sum_{c > i^2} a_4(1,c-i^2,1)$$

for every $c \geq 1$.

The equations of the theorem follow directly from these by noting the symmetry of i^2 and i^2+i

about 0 and ½ respectively, and by defining Δ appropriately.

QED

Remarks

(1) It is evident that equations (II-7) and (II-8) are recursive equations. Moreover, they are evaluated solely from the classical theory of modular forms as we shall show below.

(2) Since ϕ_4 generates the graded ring of modular forms of degree two [7] it may be too much to hope for simple recursion relationships for the other forms.

(3) Comparing the equations of the theorem with Maass' explicit formula for the primitive matrices (AI-2) in Appendix I, we see that the former are indeed much simpler to apply.

(4) Using Conjecture I with $w = 4$ and the equation of the theorem, we can determine all the ϕ_4-coefficients.

(5) Should one use the equations in (4) to generate some ϕ_4-coefficients, several checks on the data can be made from known identities. Some are used in Part II of this paper to verify Table I and are given

in Chapter IV, section D.

(6) Table II, Chapter V gives a list of ϕ_4-coefficients generated by the equations in (4); see p.82.

2. Examples of applications of the theorem

(a) We will show that the equations of Theorem 1 can be written such that the degree 2, ϕ_4-coefficients of primitive matrices are functions only of the degree 1, ϕ_4-coefficients. We give the results in the following lemmas:

Lemma 3: Let Δ be an arbitrary integer such that $\Delta \geq 0$. Then if Conjecture II is true, we have

$$\text{(II-10)} \quad a_4(1,\Delta,0) = \sum_{0\leq 2i\leq \Delta} \alpha(2i)a_4(4\Delta-8i+1)$$

where

$$\alpha(2i) = -\sum_{j^2+j\leq 2i} \alpha(2i-j^2-j)$$

and

$$\alpha(0) = 1$$

Proof: The proof will follow by induction on Δ. Clearly the conclusion is true for $\Delta = 0$. Hence inductively we will assume the conclusion holds for all

$n \leq \Delta$ and prove it holds for $\Delta+1$.

From equation (II-7), we have

$$a_4(1,\Delta+1,0) = a_4(4\Delta+5) - \sum_{\substack{i^2+i \leq \Delta+1 \\ 0<i}} a_4(1,\Delta+1-i^2-i,0).$$

Further, it follows from the inductive hypothesis and (II-10) that

$$a_4(1,\Delta+1,0) = a_4(4\Delta+5) - \sum_{\substack{i^2+i \leq \Delta+1 \\ 0<i}} \sum_{0 \leq 2j \leq \Delta+1-i^2-i} \alpha(2j) \cdot a_4(4\Delta-8(j+\frac{i^2+i}{2})+5)$$

and, by letting $k = j+\frac{i^2+i}{2}$.

$$a_4(1,\Delta+1,0) = a_4(4\Delta+5) - \sum_{\substack{i^2+i \leq \Delta+1 \\ i>0}} \sum_{0 \leq 2k \leq \Delta+1} \alpha(2k-i^2-i) \cdot a_4(4\Delta-8k+5).$$

Then, by interchanging the summations, we obtain the desired result for $\Delta+1$.

$$a_4(1,\Delta+1,0) = a_4(4\Delta+5) + \sum_{0\le 2k\le \Delta+1} [- \sum_{i^2+i\le 2k} \alpha(2k-i^2-i)]$$

$$\cdot\ a_4(4\Delta-8k+5).$$

QED

and

Lemma 4: Let Δ be an arbitrary positive real number satisfying $4\Delta\equiv 3 \pmod 4$ and define $m = \frac{4\Delta+1}{4}$. Then if Conjecture II is true, we have

$$\text{(II-11)}\quad a_4(1,m,1)= \sum_{i=1}^{m} \beta(i)a_4(4m-4i+3)$$

where

$$\beta(i) = -2 \sum_{\substack{j^2<i \\ j>0}} \beta(i-j^2)$$

and

$$\beta(1) = 2$$

Proof: The proof will follow by induction on m. First we note that as Δ ranges over the positive real numbers satisfying $4\Delta\equiv 3(\text{mod } 4)$, m ranges over all the integers ≥ 1; and the induction is valid.

By [2], $a_4(1,1,1) = 2^7 \cdot 3 \cdot 5 \cdot 7$ and by (I-12), $a_4(3) = 2^6 \cdot 3 \cdot 5 \cdot 7$, hence (II-11) is true for $m = 1$. Now assume the conclusion holds for all $n \leq m$, we will prove it holds for $m+1$.

From equation (II-8),

$$a_4(1,m+1,1) = 2[a_4(4m-1) - \sum_{\substack{i^2<m+1\\ i>0}} a_4(1,m+1-i^2,1)] .$$

From the inductive hypothesis and (II-11),

$$a_4(1,m+1,1) = 2[a_4(4m-1) - \sum_{\substack{i^2<m+1\\ i>0}} \sum_{j=1}^{m+1-i^2} \beta(j) a_4(4m-4(k^2+j)+7)]$$

and, by letting $k = i^2+j$.

$$a_4(1,m+1,1) = 2[a_4(4m-1) - \sum_{\substack{i^2<m+1\\ i>0}} \sum_{k=i^2+1}^{m+1} \beta(k-i^2) a_4(4m-4k+7)]$$

Also since $i>0$ and $0<i^2+1\leq k\leq m+1$, we have $2\leq k\leq m+1$ and $i^2\leq k-1$ or $i^2<k$. Hence we can interchange the sums in the last equation and get

$$a_4(1,m+1,1) = 2[a_4(4m-1)+\sum_{k=2}^{m+1}[-\sum_{\substack{i^2<k\\ i>0}}\beta(k-i^2)]a_4(4m-4k+7)]$$

or equivalently,

$$a_4(1,m+1,1) = \sum_{k=1}^{m+1}[-2\sum_{\substack{i^2<k\\ i>0}}\beta(k-i^2)]a_4(4m-4k+7)$$

where $\beta(1) = 2$

QED

For completeness, we tabulate a few of the $\alpha(\cdot)$ and $\beta(\cdot)$ coefficients.

i	$\alpha(2i)$	$\beta(i)$	i	$\alpha(2i)$	$\beta(i)$
0	1	--	7	-7	80
1	-1	2	8	10	-128
2	1	-4	9	-13	200
3	-2	8	10	16	-308
4	3	-16	11	-21	464
5	-4	28	12	28	-688
6	5	-48	13	-35	1008

(b) Another consequence of Conjecture II is an identity for degree 1 ϕ_4-coefficients and an identity for the elementary divisor function $\sigma_3(m) = \sum_{d|m} d^3$. These come about by coupling Witt's identity (I-15) on the first column of the trace planes with the equations of Lemmas 3 and 4. (Recall that a column on a trace plane P_α is the set of all lattice points $T = (a,c,b) \in \mathfrak{T}$ such that $(a+c) = \alpha$ and $0 \le b \le 2\sqrt{ac}$; we denote the first column as the column with $a=1$).

First we reformulate Witt's identity on the first column in the following lemma.

<u>Lemma 5</u> (<u>Witt's identity - 1st column</u>): Let $f \in (\Gamma_2, w)$ with Fourier coefficients $a_w(T)$. Then the Fourier coefficients for the first column of the trace plane P_{m+1} are related by

$$a_w(1)a_w(m) = \sum_{i^2 \le m} a_w(1,m-i^2,0) + 2 \sum_{\substack{i^2+i<m \\ 0 \le i}} a_w(1,m-i^2-i,1) \, .$$

<u>Proof</u>: By definition, the first column of P_{m+1}

consist of all matrices $T = (1,m,b)\in\mathfrak{T}$ where $|b|\le 2\sqrt{m}$. Let $T_b = (1,m,b), b\in\mathbb{Z}$ and let S_n be the unimodular matrix, $S_n = \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$. The b's decompose into two groups, $b\equiv 0 \pmod 2$ and $b\equiv 1 \pmod 2$. Taking each case separately, we get

<u>Case 1</u>: Let $b\equiv 0 \pmod 2$. i.e., $b = 2\bar{w}$, $\bar{w}\in\mathbb{Z}$ and choose $n = -b/2$. Then we can write T_b as

$$T_b[S_{-\frac{b}{2}}] = (1, (-\tfrac{b}{2})^2 - \tfrac{b^2}{2} + m, -b+b) = (1, m-\tfrac{b^2}{4}, 0).$$

By the assumption on b, we get

$$T_b[S-\tfrac{b}{2}] = (1, m-\tfrac{4\bar{w}^2}{4}, 0) = (1, m-\bar{w}^2, 0)$$

and this must be true for all $|b|\le 2\sqrt{m}$, implying $4\bar{w}^2\le 4m$ or for all $\bar{w}$ such that $\bar{w}^2\le m$.

<u>Case 2</u>: Let $b\equiv 1 \pmod 2$, i.e., $b = 2\bar{w}+1$, $\bar{w}\in\mathbb{Z}$ For the unimodular matrix S_n above with $n = \frac{1-b}{2}$, we have

$$T_b[S_{\frac{1-b}{2}}] = (1, (\tfrac{1-b}{2})^2 + (\tfrac{1-b}{2})b+m,\ (1-b)+b)$$

$$= (1, m+\tfrac{1-b^2}{4}, 1)$$

By the assumption on b, we get

$$T_b[S_{\frac{1-b}{2}}] = (1, m-\frac{1-(2\overline{w}+1)^2}{4}, 1)$$

$$= (1, m-\frac{1-4\overline{w}^2-4\overline{w}+1}{4}, 1)$$

$$= (1, m-\overline{w}^2-\overline{w}, 1)$$

and this must be true for all $|b| \le 2\sqrt{m}$ implying $4\overline{w}^2+4\overline{w}+1 \le 4m$ or for all $\overline{w}$ such that $\overline{w}^2+\overline{w} \le m-\frac{1}{4} < m$.

The invariance of the Fourier coefficients under unimodular transformation (I-6) gives the desired result.

QED

Now the coupling Lemma 5 and Lemmas 3 and 4, we get the following identity for the classical ϕ_4-coefficients. Here, we recall that $a_4(m)=240\sigma_3(m)$ by (I-12).

Lemma 6: Let $a_4(.)$ be the classical ϕ_4-coefficients. Then if Conjecture II is true, we have

$$240\ a_4(m) = \sum_{k=0}^{2m} \gamma(k) a_4(4m-2k+1)$$

where

$$\gamma(k) = \begin{cases} \sum\limits_{i^2 \le k} \alpha(k-i^2) & , \quad k \equiv 0 \pmod 2 \\ \sum\limits_{\substack{i^2+i \le k \\ 0 \le i}} 2\beta(k-i^2-i+1), & k \equiv 1 \pmod 2 \end{cases}$$

Here $\alpha(\cdot)$ and $\beta(\cdot)$ are the functions defined in Lemmas 3 and 4, respectively.

Proof: By Lemma 5, we get

(II-12)

$$240\ a_4(m) = \sum_{i^2 \le m} a_4(1,m-i^2,0)+2 \sum_{\substack{i^2+i<m \\ 0\le i}} a_4(1,m-i^2-i,1)$$

First term: Denoting the first term on the right side of (II-12) by F_1 and substituting the results of Lemma 3, we get

$$F_1 = \sum_{i^2 \le m} \sum_{0 \le 2j \le m-i^2} \alpha(2j) a_4(4m-4[i^2-2j]+1)$$

where $\alpha(.)$ is defined by the Lemma 3.

Letting $k = i^2+2j$ implies $2j = k-i^2$, so that

$$F_1 = \sum_{i^2 \le m} \sum_{i^2 \le k \le m} \alpha(k-i^2) a_4(4m-4k+1).$$

Interchanging the summations yields

$$F_1 = \sum_{k=0}^{m} \sum_{i^2 \le k} \alpha(k-i^2) a_4(4m-4k+1) .$$

Second term: Letting F_2 denote the second term in (II-12), we get by substitution of the results of Lemma 4,

$$F_2 = 2 \sum_{\substack{i^2+i<m \\ 0\le i}} \sum_{1\le j\le m-i^2-i} \beta(j)a_4(4m-4(i^2+i+j-1)-1)$$

where $\beta(\cdot)$ is defined by Lemma 4.

Letting $k = i^2+i+j-1$ implies $j = k^2-i^2-i+1$, so that

$$F_2 = 2 \sum_{\substack{i^2+i<m \\ 0\le i}} \sum_{i^2+i\le k\le m-1} \beta(k-i^2-i+1)a_4(4m-4k-1),$$

Interchanging summations implies

$$F_2 = 2 \sum_{k=0}^{m-1} \sum_{\substack{i^2+i\le k \\ i\ge 0}} \beta(k-i^2-i+1)a_4(4m-4k-1)$$

Combining the first and second terms, we get

$$240\, a_4(m) = F_1 + F_2$$

$$= \sum_{k=0}^{m} \Big[\sum_{i^2 \le k} \alpha(k-i^2) \Big] a_4(4m-4k+1)$$

$$+ \sum_{k=0}^{m-1} \Big[2 \sum_{\substack{i^2+i \le k \\ i \ge 0}} \beta(k-i^2-i+1) \Big] a_4(4m-4k-1),$$

Finally, noting that

$$4m-4k-1 = 4m-2(2k+1)+1$$

and

$$4m-4k+1 = 4m-2(2k)+1$$

we get the results of the lemma.

QED

This lemma yields immediately the following identity for the elementary divisor functions σ_3 :

$$\text{(II-13)} \quad 240\, \sigma_3(m) = \sum_{k=0}^{2m} \gamma(k) \sigma_3(4m-2k+1)$$

where $\gamma(\cdot)$ is the function defined by Lemma 6. Here again, we point out that (II-13) holds if Conjecture II is true. Investigations into Dickson's classic reference [1] has not produced evidence that this identity (if it is true) is known. We tabulate a few of the $\gamma(.)$ coefficients.

i	$\gamma(i)$	i	$\gamma(i)$
0	1	7	-40
1	4	8	3
2	2	9	72
3	-8	10	2
4	-1	11	-128
5	20	12	-4
6	-2	13	220

PART TWO

TABLE OF FOURIER COEFFICIENTS

III. The Differential Operators and Recursive Equations

In the first part of this paper, we have stated two conjectures concerning the Fourier coefficients of degree two modular forms. We are aware that new conjectures are often met with skepticism when introduced without a sufficient base of examples to render them credible. In order to try to mitigate this, we have generated a large set of Fourier coefficients for selected modular forms with which to substantiate the conjectures. This, admittedly, only adds credence to the conjectures and can afford no insights for proof. It is the purpose of this chapter and the next two to present these coefficients and the method used in generating them.

Igusa proved in his fundamental structure theorem [2] that the graded ring of modular forms of even weight is generated by the Eisenstein series of weights 4, 6, 10, and 12. Following Igusa, the Eisenstein series of weight 10 and 12 can be replaced by the cusp forms,

$$\text{(III-1)} \qquad \chi_{10} = -43867 \cdot 2^{-12} \cdot 3^{-2} \cdot 7^{-1} \cdot 53^{-1} (\phi_4\phi_6 - \phi_{10})$$

and

(III-2)

$$\chi_{12} = 131\cdot 593 \cdot 2^{-13}\cdot 3^{-7}\cdot 5^{-3}\cdot 7^{-2}\cdot 337^{-1}(3^2 7^2 \phi_4^3 + 2\cdot 5^3 \phi_6^2 - 691\phi_{12}).$$

Although the algebraically independent modular forms ϕ_4, ϕ_6, χ_{10}, and χ_{12} generate the graded ring, H. Resnikoff [7] has shown that ϕ_4 and its image under certain non-linear differential operators generate the same ring. Therefore, the forms $\phi_4, \phi_6, \chi_{10}, \chi_{12}$ are not algebraically dependent, but are differentiably dependent. That is, in the sense that there is a non-trivial polynomial with complex coefficients in the functions $\phi_4, \phi_6, \chi_{10}, \chi_{12}$, and their derivatives which vanishes. Moreover, the conclusion states that complete information about the graded ring of forms lies in ϕ_4 and hence in its Fourier coefficients.

Our main concern in this chapter shall be to show that from this same differential operator theory recursive equations for the $\phi_4, \phi_6, \chi_{10}$, and χ_{12} coefficients can be derived. Then in Chapter IV, we will develope an effective algorithm for evaluating these equations. The resulting table of coefficients is given in Chapter V.

1. The differential operators are defined as follows for $f\in(\Gamma,w)$:

$$D^n f = f^{2n+\frac{\frac{3}{2}-n}{w}}\,\partial_Z^n f^{\frac{n-\frac{3}{2}}{w}}$$

where

$$\partial_Z = \frac{\partial^2}{\partial z_1 \partial z_2} - \frac{1}{4}\,\frac{\partial^2}{\partial z_{12} \partial z_{12}}$$

Furthermore, we have

Proposition 5: (1) $D^n:(\Gamma,w)\rightarrow(\Gamma,n(2w+2))$
(2) $D^n\,f$ is a cusp form.

[7] shows that $D^n f$ can be expressed in the form

$$\text{(III-3)} \qquad D^n f = \sum_{k=1}^{2n} A_k(w,n) f^{2n-k}\partial_Z^n f^k$$

where

$$A_k(w,n) = \prod_{\substack{j=0\\ j\neq k}}^{2n} \frac{\left(\frac{\frac{3}{2}-n}{w}-j\right)}{k-j} .$$

Applying the differential operators to the generators of the graded ring of modular forms, we can state the results of [7].

Proposition 6.

(i) $D^1\phi_4 = 2^6\cdot 3^2\cdot 5\cdot 7\pi^2\chi_{10}$

(ii) $D^1\phi_6 = 2^6\cdot 3^2\cdot 7\cdot 11\pi^2\phi_4\chi_{10}$

(iii) $D^1\chi_{10} = 3^2/2^4\cdot 5^2\pi^2\chi_{10}\chi_{12}$

(iv) $D^2\chi_{10} = 2^6\cdot 3^2\cdot 5\cdot \phi_4\chi_{10}^4 - 3^2\cdot 2^4\cdot 5^2(\chi_{10}\chi_{12})^2$

It is easily seen that

$$\partial_Z^n\, e(TZ) = (2\pi i)^{2n}|T|^n e(TZ).$$

This and equation (I-9) lead to the following:

Lemma 7: For $f\in(\Gamma,w)$ with Fourier coefficients $a_4(\cdot)$,

(i) $\partial_Z^n f^k_{(T)}(Z) = (2\pi i)^{2n}|T|^n f_{(T)}(Z)$

(ii) $\partial_Z^n f^k(Z) = (2\pi i)^{2n} \sum_{\substack{(D)\geq 0\\ D=\sum_1^k T_i}} \prod_1^k a_w(T_i)|D|^n C_D f_{(D)}(Z)$

(iii) $f^{\ell}(Z)\partial_Z^n f^k(Z) = (2\pi i)^{2n} \sum_{\substack{(F)\geq 0\\ F=\sum_1^{\ell+k} T_i}} \prod_1^{\ell+k} a_w(T_i)$

$\cdot \left|\sum_{\ell+1}^{\ell+k} T_j\right|^n \cdot C_F f_{(F)}(Z)$

Proof: (i) $\partial_Z^n f_T(Z) = \partial_Z^n[\sum_{T_1 \in (T)} e(T_1 Z)]$

$$= \sum_{T_1 \in (T)} \partial^n e(T_1 Z)$$

$$= \sum_{T_1 \in (T)} (2\pi i)^{2n} |T_1|^n e(T_1 Z)$$

$$= (2\pi i)^{2n} |T|^n f_{(T)}(Z) \quad .$$

(ii) $\partial_Z^n f(Z)^k = \partial_Z^n[\sum_{(D) \geq 0} \prod_1^k a_w(T_i) C_D f_{(D)}(Z)]$

$$= \sum_{(D) \geq 0} \prod_1^k a_w(T_i) C_D [\partial_Z^n f_{(D)}(Z)]$$

$$= (2\pi i)^{2n} \sum_{(D) \geq 0} \prod_1^k a_w(T_i) |D|^n C_D f_{(D)}(Z) \quad .$$

(iii) $f^{\ell}(Z) \partial_Z^n f^k(Z) = [\sum_{(D) \geq 0} \prod_1^{\ell} a_w(T_i) C_D f_{(D)}(Z)]$

$$\cdot [(2\pi i)^{2n} \sum_{G \geq 0} \prod_1^k a_w(T_i) |G|^n C_G f_{(G)}(Z)]$$

$$= (2\pi i)^{2n} \sum_{\substack{F \geq 0 \\ F = \sum_1^{\ell+k} T_i}} \prod_1^{\ell+k} a_w(T_i) \left| \sum_{\ell+1}^{\ell+k} T_j \right|^n \cdot C_F f_{(F)}(Z) .$$

QED

Hence equation (III-3) with this lemma becomes

$$D^n f = \sum_{k=1}^{2n} A_k(w,n)[(2\pi i)^{2n} \sum_{\substack{F\geq 0\\ F=\sum_1^{\ell} T_i}} (\prod_{i=1}^{\ell} a_w(T_i) | \sum_{j=m}^{\ell} T_j |^n) C_F f_{(F)}(Z)]$$

(III-4)

$$= \sum_{\substack{F\geq 0\\ F=\sum_1^{2n} T_i}} [(2\pi i)^{2n} \sum_{k=1}^{2n} (A_k(w,n) \cdot | \sum_{j=m}^{2n} T_j |) \prod_{i=1}^{2n} a_w(T_i)] C_F f_{(F)}(Z)$$

where $A_k(w,m) = \prod_{\substack{j=0\\ j\neq k}}^{2n} \left(\frac{w-j}{k-j} \right)^{\frac{3}{2}-n}$, $\ell = 2n$, and $m = 2n-k+1$.

For the case $n=1$, we have

Lemma 8:

$$D^1 f = A_1(w,1) f \partial f + A_2(w,1) \partial f^2$$

$$= \sum_{\substack{(D)\geq 0\\ D=T_1+T_2}} [(2\pi i)^2 (A_1(w,1)|T_2| + A_2(w,1)|D|) a_w(T_1) a_w^{\gamma}(T_2)] \cdot C_D f_{(D)}(Z)$$

where $A_1(w,1) = (\frac{8w-2}{8w^2})$ and $A_2(w,1) = (\frac{1-2w}{8w^2})$.

2. Resnikoff [8] pointed out that by substitution of the Fourier expansion of $\phi_4, \phi_6, \chi_{10}$, and χ_{12} into the differential equations of Proposition 6 one could produce non-linear recursion formulae for the coefficients of these modular forms. Using equations (III-4) and lemma 8 we record these recursion relationships in the following propositions.

Proposition 7: Let $a_4(T)$ and $c_{10}(T)$ be the Fourier coefficients of ϕ_4 and χ_{10} respectively. Then for every $T \geq 0$,

$$2^{11} \cdot 3^2 \cdot 5 \cdot 7 c_{10}(T) = \sum_{\substack{T=T_1+T_2 \\ T_i \geq 0}} \{7|T| - 30|T_1|\} a_4(T_1) a_4(T_2). \tag{III-5}$$

Proof: Using equation (i) of proposition with χ_{10} expressed in its Fourier expansion and lemma 8 to express $D^1\phi_4$ in its series expansions, we have by uniqueness of Fourier expansions the result.

QED

Proposition 8: Let $a_4(T)$ and $c_{10}(T)$ be the Fourier coefficients of ϕ_4 and χ_{10} respectively.

Then for every $T \geq 0$,

$$\sum_{\substack{T=T_1+T_2+T_3+T_4 \\ T_i>0}} \{2^2 \cdot 3^6 \cdot 41 \cdot 61|T_1|^2 - 2 \cdot 3^8 \cdot 7 \cdot 61|T_1+T_2|^2$$

$$+2^2 \cdot 3^7 \cdot 7 \cdot 41\ |T_1+T_2+T_3|^2 - 3^3 \cdot 7 \cdot 41 \cdot 61|T_1$$

$$+T_2+T_3+T_4|^2 + 2^3 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13^2 \cdot 21|T_1||T_3|$$

(III-6) $$-2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31|T_1||T_3+T_4|$$

$$-2^3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31|T_1+T_2||T_3|$$

$$+2 \cdot 7 \cdot 11 \cdot 19^2 \cdot 31|T_1+T_2||T_3+T_4|\}$$

$$c_{10}(T_1)c_{10}(T_2)c_{10}(T_3)c_{10}(T_4)$$

$$= \sum_{\substack{T=T_0+T_1+T_2+T_3+T_4 \\ T_0\geq 0, T_i>0 \\ \text{for } i>0}} \{2^2 \cdot 3^2 \cdot 5 \cdot 59\} a_4(T_0)c_{10}(T_1)c_{10}(T_2)c_{10}(T_3)c_{10}(T_4)$$

Proof: Using equation (III-2) to expand $D^2\chi_{10}$ in its Fourier series and equation (I-9) to express the products of $\phi_4\chi_{10}^4$ and $(\chi_{10}\chi_{12})^2$ in series, equation iv of Proposition 6 yields the conclusion of the theorem.

QED

Proposition 9: Let $a_6(T)$ and $c_{12}(T)$ be the Fourier coefficients of ϕ_6 and χ_{12} respectively and let $a_4(T)$ and $c_{10}(T)$ be defined as above. Then for every $T\geq 0$,

$$\text{(III-7)} \qquad \sum_{\substack{T=T_1+T_2 \\ T_i\geq 0}} [\{11|T|-46|T_2|\}a_6(T_1)a_6(T_2) - 2^9\cdot 3^4\cdot 7\cdot 11\ a_4(T_1)c_{10}(T_2)]=0$$

and (III-8)

$$\sum_{\substack{T=T_1+T_2 \\ T_i>0}} [\{19|T|-78|T_2|\}c_{10}(T_1)c_{10}(T_2) - \frac{3^2}{2} c_{10}(T_1)c_{12}(T_2)]=0$$

We remark that while the relationships of the propositions are well defined, some further investigation will be necessary to show that they recursively determine the $\phi_4, \phi_6, \chi_{10}$ and χ_{12} coefficients. This is the purpose of the next chapter.

IV. Reduction of the Problem and the Algorithm for Computation

The purpose of this chapter is to demonstrate that an effective algorithm exists for calculating the coefficients for $\phi_4, \phi_6, \chi_{10}$ and χ_{12} using the relationships of the last chapter.

A. Preliminary Remarks

(1) All the Fourier coefficients $a_w(T)$ of the Eisenstein series, ϕ_w, for $w = 4,6,8,10$ and which satisfy $\sigma(T) \leq 2$ have been calculated by Igusa [2]. For reference, we tabulate his results here.

w	$a_w\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
4	$2^5 \cdot 3^3 \cdot 5 \cdot 7$
6	$2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
8	$2^6 \cdot 3^2 \cdot 5 \cdot 61$
10	$2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 277 \cdot 43867^{-1}$

w	$a_w\begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}$
4	$2^7 \cdot 3 \cdot 5 \cdot 7$
6	$2^6 \cdot 3^2 \cdot 7 \cdot 11$
8	$2^8 \cdot 3 \cdot 5 \cdot 7$
10	$2^6 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 809 \cdot 43867^{-1}$

(2) The action of the Siegel Operator (Chapter I-D) gives us the formula for $a_w(T)$ when $T=\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$. Rewriting the result here

(IV-2) $a_w(T) = a_w(m) = a_w(1)\sigma_{w-1}(m).$

$a_w(1)=240, -504, -264, \frac{54,600}{691}$ for $w = 4,6,10$ and 12. $\sigma_k(\cdot)$ is the elementary divisor function.

(3) Using the known coefficients $a_w\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for ϕ_4, ϕ_6, and ϕ_{10} given by (IV-1), and expanding χ_{10} in its Fourier series, we can determine from (III-1) that $c_{10}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}$. Witt's identity (I-15) then gives us $c_{10}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = \frac{1}{4}$. Thus all the χ_{10} Fourier coefficients are known for $\sigma(T) \le 2$. Further, since χ_{10} is a cusp form, $c_{10}(T) = 0$ for all $T \in \mathfrak{J}$ such that $|T|=0$ (Definition 5).

(4) From Chapter I-F, we know that we need only evaluate the Fourier coefficients of modular forms for lattice points in the reduced triangles of each trace plane (fig. 2). That is, on the set

$R = \{T=(a,c,b) = T \in \mathfrak{J} \text{ and } 0 \le b \le a \le c\}.$

Moreover, Proposition 3 gives us an algorithm for reducing any arbitrary matrix $T \in \mathfrak{T}$ to an element of R.

B. Decompositions of Matrices

Even though the remarks of the previous section tell us that we need only evaluate the coefficients for the lattice points of the triangles of a given trace plane, the question still remains as to which trace plane and what lattice points within the reduced triangle to consider first. To answer this question, we must look at the recursive equations to be solved. Namely the equations of Propositions 7,8, and 9.

One first of all notices that the recursive equations consist of terms of the Fourier coefficients of matrices associated with a decomposition of a positive definite matrix T. If we are ever going to hope to solve for a particular coefficient of a given matrix $A \in \mathfrak{T}$, this matrix T must contain A as one of the terms in a decomposition of T. We also see that the decompositions of T which we will require have the general form,

$$T = \sum_{i=0}^{m} T_i + \sum_{j=1}^{n} S_j$$

where $T_i > 0 (0 \le i \le m)$ and $S_j \ge 0$ $(0 \le j \le n)$. Therefore, if we define the augumented matrix of A to be

$$T = A + m\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} + n \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we have most certainly forced the coefficient of A to appear at least once in the evaluation of the equation for T.

We prove the following lemma.

Lemma 10: Let T be a matrix of the form

$$\text{(IV-3)} \qquad T = A + m\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} + n \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where $A = (a_1, a_2, a_3) > 0$ is an element of $\mathfrak{J}$. Then for any decomposition of T of the form

$$\text{(IV-4)} \qquad T = \sum_{i=0}^{m} T_i + \sum_{j=0}^{n} S_j$$

where

$$T_i = (t_{i1}, t_{i2}, t_{i3}) > 0 \qquad (0 \le i \le m)$$

and

$$S_j = (s_{j1}, s_{j2}, s_{j3}) \ge 0 \qquad (0 \le j \le n) \text{ ,}$$

we have

(i) $\sigma(A)>\sigma(S_j)$ for every $0\le j\le n$,

(ii) $\sigma(A)\ge\sigma(T_i)$ for every $0\le i\le m$,

and (iii) if $\sigma(A)=\sigma(T_{\hat{i}})$ for some $0\le\hat{i}\le m$,

then $t_{\hat{i}3}\ge a_3$.

<u>Proof</u>: Let T be an arbitrary positive definite matrix defined by (IV-3). We recall that positive definite matrices must have strictly positive diagonal entries, and semi-positive definite matrices must have either positive or zero entries on the diagonal. Then for any decomposition of T defined by equation (IV-4), we must have

$$a_j+m = \sum_{i=0}^{m} t_{ij} + \sum_{i=0}^{n} s_{ij},\ j=1,2$$

where $a_j\ge1$, $t_{ij}\ge1$, and $s_{ij}\ge0$.

(i) Suppose that there exists an $0\le\hat{i}\le n$ such that $s_{\hat{i}j}\ge a_j\ge1$, $j=1,2$. Then for a decomposition we must have

$$a_j+m\ge m+1+s_{\hat{i}j}\ge m+1+a_j$$

which is absurd. Hence we get

$$s_{ij}<a_j \quad \text{for every} \quad 0\le i\le n,\ j=1,2,$$

and therefore

$$\sigma(S_i) = s_{i1}+s_{i2}<a_1+a_2 = \sigma(A)$$

for every $0\le i\le n$.

(ii) Suppose that there exists an $0\le\hat{i}\le m$ such that $t_{\hat{i}j}>a_j\ge 1$ $j=1,2$. Then assuming the decomposition to be valid, we get

$$a_j+m\ge t_{\hat{i}j}+m>a_j+m$$

which is impossible. Therefore we must have

$$t_{ij}\le a_j \quad \text{for every} \quad 0\le i\le m,\ j=1,2.$$

From this we have

$$\sigma(T_i) = t_{i1}+t_{i2}\le a_1+a_2 = \sigma(A)$$

for every $0\le i\le m$.

(iii) Note that by argument similar to the above $\sigma(A) = \sigma(T_i)$ can hold only for one matrix $T_{\hat{i}}, 0\le\hat{i}\le m$

in which case, $t_{\hat{i}1}=a_1$ and $t_{\hat{i}2}=a_2$. So assuming $\sigma(A)=\sigma(T_{\hat{i}})$, $0\le\hat{i}\le m$, we get $t_{ij}=1$ for every $0\le i(\ne\hat{i})\le m$; and $s_{ij}=0$ for every $0\le i\le n$, $j=1,2$. Hence for $T_i>0(0\le i(\ne\hat{i})\le m)$, we must have $|T|=\begin{vmatrix}1 & \frac{t_{i3}}{2}\\ \frac{t_{i3}}{2} & 1\end{vmatrix}>0$ implying

$t_{i3}^2<2$ or that $|t_{i3}|\le 1$. For $S_i(0\le i\le n)$, we must have $\begin{vmatrix}0 & \frac{s_{i3}}{2}\\ \frac{s_{i3}}{2} & 0\end{vmatrix}\ge 0$ implying $-s_{i3}^2\ge 0$ such that $s_{i3}=0$

for every $0\le i\le n$. In particular, S_i equals the null matrix for every $0\le i\le n$. Now assume that (iii) is not true, then $t_{\hat{i}3}<a_3$. For the decomposition to be valid (using the above conclusions), we must have

$$a_3+m=t_{13}+\ldots+t_{\hat{i}3}+\ldots+t_{m3}\le t_{\hat{i}3}+m<a_3+m$$

a contradiction. Hence (iii).

QED

This lemma shows that by introducing a given matrix $A=(a_1,a_2,a_3)\in\mathfrak{T}$ into the recursion equations by means of (IV-3), we can obtain an equation in the

Fourier coefficient associated with A. Further, if the coefficients for all lattice points $T = (t_1, t_2, t_3) \in \mathcal{T}$ with $\sigma(T) = \sigma(A)$ and $t_3 > a_3$ and for all lattice points on previous trace planes are known, the resulting equation can be solved for the A-coefficient. This follows directly from the lemma since the matrices of the decomposition of T, T_i and S_j of equation (IV-4), all occur on previous trace planes or on the same trace plane as A with $t_3 \geq a_3$.

In summary, the answer to the question at the head of this section is that we may begin on any trace plane with any lattice point $A = (a_1, a_2, a_3)$ where the coefficients associated with lattice points $T = (t_1, t_2, t_3)$ on previous trace planes and on the same trace plane with $t_3 \geq a_3$ are known. Further, we know that lattice points outside the reduced triangle are equivalent to lattice points on previous trace planes (Corollary 1). Hence with respect to figure 2, within each trace plane P_α we are at liberty to begin with any lattice point $A = (a_1, a_2, a_3)$ with $(a_1 + a_2) = \alpha$ and $a_1 \leq a_2 \leq 2\sqrt{a_1 a_2}$. Once beginning on a given column (defined by the set of all lattice points T such that $(a_1 + a_2) = \alpha$ and $0 \leq a_3 \leq 2\sqrt{a_1 a_2}$, one must take the lattice points of that column in the reduced triangle in descending order of a_3.

C. The Computation

Using the above information, we are now in a position to see that with the possible exception of finitely many matrices $T \in \mathfrak{T}$, the Fourier coefficients of $\phi_4, \phi_6, \chi_{10}$, and χ_{12} are computable from the recursive equations.

The following procedure was used in evaluating the coefficients $a_4(T)$ of ϕ_4 and $c_{10}(T)$ of χ_{10} on a Control Data Corporation 3800 Computer. Time requirements and costs necessary to complete this task are given in Appendix II.

1. For initialization of the recursion, we assumed the values of $a_4(T)$ and $c_{10}(T)$ for $\sigma(T) \leq 2$ given by (IV-1) and by remark of section A, respectively.

2. We set all values of $c_{10}(T) = 0$ for $|T| = 0$ by remark 3 of section A, and determined all values of $a_4(T)$ for $T = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$ by equation (IV-2).

3. Using the method summarized at the end of section B, a matrix $T_1 \in R$ was selected on the trace plane P_3. Then the matrix $T = T_1 + 3\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$ was formed to be decomposed as in (IV-4).

4. An (unsophisticated) method of determining all decompositions of T and evaluating equation (III-6) for $a_4(T_1)$ was mechanized.

5. Letting $T = T_1$ in equation (III-5) and determining the needed decompositions of T as in 4, we evaluated (III-5) for $c_{10}(T_1)$.

6. We proceeded calculating the coefficients down columns of the reduced triangles in trace plane 3 with steps 3,4 and 5. At each determination of the coefficient $a_4(T_1)$, we used step 5 to determine $c_{10}(T_1)$ for the same T_1.

7. Observing the natural ordering of the trace planes, we repeated step 6 to the other trace planes.

A similar mechanization was made for the evaluation of the ϕ_6 and χ_{12} coefficients using equations (III-7) and (III-8), respectively.

D. <u>Checks on the Computations</u>

The following checks we used to verify the coefficients obtained from the computer program which used the above procedure. The primary checks consist of programming verification and Witt's identity; while the secondary checks make use of the results of Maass [6].

Primary Checks:

1. Using the recursive relationships, some initial unknown coefficients were calculated independent of the computer program. A comparison of these coefficients to the computer calculated coefficients was made for program verification.

2. Since the recursive equations do not depend on the invariance of the coefficients under unimodular transformations, a few coefficients for lattice points on trace plane three and four not in R were calculated. These were then compared to the coefficients in R.

3. Upon completion of each column in a given trace plane, an application of Witt's identity (I-15) was made.

Secondary Checks:

1. By appendix I, primitive matrices with the same determinant have equal Fourier coefficients. All coefficients for primitive matrices were calculated for ϕ_4 and χ_{10}.

2. Selected imprimitive matrices were substituted into equation (AI-3) of Appendix I. This

resulted in an equation for the coefficients of these imprimitive matrices in terms of primitive matrix coefficients. Then using the computer calculated primitive matrix coefficients, the former coefficients were evaluated.

3. The congruence condition of equation (AI-4) of Appendix I was applied to applicable lattice points.

V. The Calculated Coefficients of $\phi_4, \phi_6, \chi_{10}$ and χ_{12}

1. Using the algorithm and checks of the previous chapter, the $\phi_4, \phi_6, \chi_{10}$, and χ_{12} Fourier coefficients were calculated and tabulated in Table I. Since primitive matrices with equal determinants have equal Fourier coefficients, the table includes only the primitive representatives $(1,\Delta,0)$ and $(1,\frac{4\Delta+1}{4},1)$.

The ϕ_4 and χ_{10} coefficients were calculated and verified for almost all matrices $T \in \mathfrak{J}$ with $|T| \leq 12$ (Table I-a). These coefficients complete the knowledge of the ϕ_4 and χ_{10} Fourier coefficients for all lattice points on trace planes $\alpha \leq 7$ and parts of trace planes 8 through 13 (figure 2). An inspection of the time required to generate each of these coefficients (Appendix II) indicates why no additional coefficients were calculated.

Similarly, the ϕ_6 and χ_{12} coefficients for matrices $T \in \mathfrak{J}$ with $|T| \leq 4$ were calculated (Table 1-b). These calculations used the recursive equations of (III-7) and (III-8), respectively. Included are all coefficients of trace planes $\alpha \leq 4$ along with two coefficients of trace plane 5. As before, the table has been limited only by the computation time required for each coefficient.

It is important to note that the Fourier coefficients of Table I represent the first known table of any length for Siegel modular forms. Moreover, the two conjectures of Part One are found to be true for the coefficients given in this table. This, of course, adds a great deal of support to the conjectures.

2. Table II contains a set of conjectured ϕ_4-coefficients. The primitive matrix coefficients were calculated using the conjectured equations of Lemmas 3 and 4 (Chapter II), and the imprimitive matrix coefficients by a special case of the Conjecture I. The table extends Table I to lattice points on all trace planes $\alpha \leq 10$ and to parts of trace planes 11 through 26. This set includes matrices $T \in \mathcal{J}$ such that $|T| \leq 25$. All applicable checks of Chapter IV-section D have been applied to these additional coefficients and have been found to hold.

Table I

Fourier Coefficients for $\phi_4, \phi_6, \chi_{10}$, and χ_{12}

a. Coefficients of ϕ_4 and χ_{10}

		ϕ_4	χ_{10}
4|T|	T	$a_4(T)$	$4c_{10}(T)$
3	(1,1,1)	13,440.	-1.
4	(1,1,0)	30,240.	2.
7	(1,2,1)	138,240.	16.
8	(1,2,0)	181,440.	-36.
11	(1,3,1)	362,880.	-99.
12	(1,3,0)	497,280.	272.
12	(2,2,2)	604,800.	-240.
15	(1,4,1)	967,680.	240.
16	(1,4,0)	997,920.	-1,056.
16	(2,2,0)	1,239,840.	-32.
19	(1,5,1)	1,330,560.	253.
20	(1,5,0)	1,814,400.	1,800.
23	(1,6,1)	2,903,040.	-2,736.
24	(1,6,0)	2,782,080.	1,464.
27	(1,7,1)	3,279,360.	4,284.
27	(3,3,3)	3,642,240.	15,399.
28	(1,7,0)	4,008,960.	-12,544.

Table I-a (continued)

4|T|	T	ϕ_4 $a_4(T)$	χ_{10} $4c_{10}(T)$
28	(2,4,2)	5,114,880.	-4,352.
31	(1,8,1)	5,806,080.	6,816.
32	(1,8,0)	5,987,520.	19,008.
32	(2,4,0)	7,439,040.	576.
35	(1,9,1)	6,531,840.	-27,270.
36	(1,9,0)	7,650,720.	4,554.
36	(3,3,0)	8,467,200.	43,920.
39	(1,10,1)	10,644,480.	6,864.
40	(1,10,0)	9,555,840.	-39,880.
43	(1,11,1)	10,039,680.	66,013.
44	(1,11,0)	13,426,560	26,928.
44	(2,6,2)	16,329,600	-23,760.
47	(1,12,1)	17,418,240.	-44,064.
48	(4,4,4)	20,818,560.	-135,424.
48	(1,12,0)	15,980,160.	-12,544.
48	(2,6,0)	19,958,400.	-126,720.
51	(1,13,1)	16,208,640.	108,102.

Table I - (continued)

b. Coefficients of ϕ_6 and χ_{12}

4|T|	T	ϕ_6 $a_6(T)$	χ_{12} $12\cdot c_{12}(T)$
3	(1,1,1)	44,352.	1.
4	(1,1,0)	166,320.	10.
7	(1,2,1)	2,128,896.	-88.
8	(1,2,0)	3,792,096.	-132.
11	(1,3,1)	15,422,400.	1275.
12	(1,3,0)	23,462,208.	736.
12	(2,2,2)	24,881,472.	2784.
15	(1,4,1)	65,995,776.	
16	(1,4,0)	85,322,160.	
16	(2,2,0)	90,644,400.	

Table II

Conjectured Fourier Coefficients for ϕ_4

$4\|T\|$	T	$a_4^{\phi_4}(T)$
52	(1,13,0)	18,264,960.
55	(1,14,1)	24,192,000.
56	(1,14,0)	23,950,080.
59	(1,15,1)	24,312,960.
60	(1,15,0)	28,062,720.
60	(2, 8,2)	35,804,160.
60	(4, 4,1)	35,804,160.
63	(1,16,1)	34,974,720.
63	(3, 6,3)	38,707,200.
64	(1,16,0)	31,963,680.
64	(2, 8,0)	39,947,040.
64	(4, 4,0)	41,882,400.
67	(1,17,1)	30,360,960.
68	(1,17,0)	38,465,280.
71	(1,18,1)	49,351,680.
72	(1,18,0)	42,638,400.
72	(3, 6,0)	47,537,280.
75	(1,19,1)	42,349,440.
75	(5, 5,5)	44,029,440.

Table II - (continued)

$4\|T\|$	T	$a_4^{\phi_4}(T)$
76	(1,19,0)	49,230,720.
76	(2,10,1)	59,875,200.
79	(1,20,1)	59,996,160.
80	(1,20,0)	59,875,200.
80	(2,10,0)	74,390,400.
80	(4, 6,4)	74,390,400.
83	(1,21,1)	56,246,400.
84	(1,21,0)	63,624,960.
87	(1,22,1)	78,382,080.
88	(1,22,0)	67,616,640.
91	(1,23,1)	66,528,000.
92	(1,23,0)	84,188,160.
92	(2,12,2)	107,412,480.
92	(4, 6,2)	107,412,480.
95	(1,24,1)	101,606,400.
96	(1,24,0)	91,808,640.
96	(2,12,0)	114,065,280.
96	(4, 6,0)	114,065,280.
99	(1,25,1)	85,276,800.
99	(3, 9,3)	95,074,560.
100	(1,25,0)	93,774,240.
100	(5, 5,0)	97,554,240.

Appendix I: Siegel's and Maass' Results

1. In [10], C. L. Siegel proved the following equation for the Fourier coefficients, $a_w(T)$, of the general Eisenstein series of degree n and weight w. For $T>0$,

$$\text{(AI-1)} \qquad a_w(T)=(-1)^{\frac{nw}{2}}\, 2^{n(w-\frac{n-1}{2})} \prod_{k=0}^{n-1} \frac{\pi^{\frac{w-k}{2}}}{\Gamma(w-\frac{k}{2})} \cdot |T|^{w-\frac{n+1}{2}} \prod_p S_p(T)$$

where p ranges through the prime numbers, and

$$S_p(T)=\sum_{R_p \bmod 1} e^{-2\pi i\sigma(TR_p)}(v(R_p))^{-w}$$

is the p-adic density. Here the sum is over a complete system (mod 1) of different n-rowed symmetric rational matrices R_p which have a power of p as a denominator. $v(R_p)$ equals the product of the divisors of R_p (Definition 8).

In order to calculate the coefficients using this formula, one has to calculate the p-adic densities for all p including $p=2$. In some cases, the Siegel operator identity (I-11) may be used to go around these tedious calculations. For $n=2$, this was the method used by J. Igusa in [2] to calculate a few of the coefficients. His table in reproduced in Chapter IV, section A of this paper.

2. For the degree two Eisenstein series, H. Maass ([6]-Satz 1) expanded the p-adic densities in (AI-1) to obtain an expression for the coefficients associated with primitive matrices (Definition 9). Letting ϕ_w be an Eisenstein series in (Γ_2,w) with Fourier coefficients $a_w(T)$, Maass' result is stated as follows.

<u>Proposition</u> <u>A1</u> <u>(Primitive</u> <u>Matrix)</u>: For $w\geq 2$ and $T(>0)$ primitive,

$$\text{(AI-2)}\quad a_w(T)=\frac{-4w}{B_w\, B_{2w-2}}\,\frac{1}{|d|}\sum_{q=1}^{|d|-1}\left(\frac{d}{q}\right)\cdot(q+|d|B)^{w-1}b_w(T)$$

with

$$b_w(T)=\left(\frac{|2T|}{|d|}\right)^{w-3/2}\prod_{p\,|\,2|2T|}\left\{\left(1-\left(\frac{d}{p}\right)p^{1-w}\right)\circ\sum_{\mu=0}^{j}p^{\mu(3-2w)}+\left(\frac{d}{p}\right)^2 p^{(j+1)(3+2w)}\right\}$$

Here $d=d(T)=$ the discriminant of the imaginary quadratic field associated with $\sqrt{-|2T|}$, $(\frac{d}{\cdot})$ is the Legendre symbol, p^b is the highest power of p dividing $|2T|$, and

$$j = \begin{cases} [\frac{b-1}{2}], & p>2 \\ [\frac{b-2}{2}], & p=2 \end{cases}$$

(greatest integer). The polynominal $(x+B)^{w-1}$ is evaluated by the identification of B^v with B_v - the v^{th} Bernoulli number (defined by equation (I-13)).

We note, as did Maass, that for a primitive matrix T, equation (AI-2) yields a value for $a_w(T)$ which is dependent only on the determinant of T. Hence we conclude that every primitive matrix with the same determinant has the same Fourier coefficient.

3. (a) From the Hecke operator theory applied to Eisenstein series of degree 2, Maass obtained the following identity for $a_w(T)$ ([6]-equation 28). For $T(>0)$ imprimitive and $w\geq 2$,

$$\text{(AI-3)}\quad a_w(T)=(1+p^{w-1})(1+p^{w-2})a_w(\tfrac{T}{p})-p^{2w-3}a_w(\tfrac{T}{p^2})$$

$$-p^{w-2}a_w(\tfrac{1}{p^2}T[\begin{smallmatrix}1&0\\0&p\end{smallmatrix}])$$

$$-\sum_{u=0}^{p-1}p^{w-2}a_w(\tfrac{1}{p^2}T[\begin{smallmatrix}u&p\\1&0\end{smallmatrix}])$$

where p is an arbitrary prime number dividing T. Terms in the equation are included with the convention that the argument of $a_w(\cdot)$ must be an element of $\mathfrak{J}$={semi-integral, semi-positive definite matrices}.

In the same paper ([6]-Satz 2),Maass attempted to show that a simplified version of equation (AI-3) was recursive for the imprimitive matrix coefficients and that $a_w(T)$ for $T=(t_1,t_2,t_3)$ was a function only of the parameters $e(T)$=g.c.d. (t_1,t_2,t_3) and $D(T)=|2T|/e^2(T)$. Unfortunately, we have found an error in his paper which invalidates his simplified recursive equation and leaves the dependence of $a_w(T)$ on $e(T)$ and $D(T)$ an open question.

In what follows, we will be able to partly correct the work of Maass by concluding that equation (AI-3) is a recursive equation for the imprimitive matrix coefficients in terms of primitive matrix coefficients. However, we will not be able to simplify equation (AI-3) nor be able to obtain the conclusion that $a_w(T)$ is a function only of $e(T)$ and $D(T)$. Nevertheless we still believe this last statement to be correct and single it out as

Maass' Conjecture: For degree two Eisenstein series with Fourier coefficients $a_w(T)$, we have

$$\text{(AI-4)} \qquad a_w(T)=\alpha_w(e(T),D(T))$$

where $\alpha_w(\cdot,\cdot)$ is a function only of $e(T)$ and $D(T)$.

(b) Before proving the assertion about equation (AI-3), we note that for $T\in\mathfrak{J}$, $e(T)$ has a standard prime factorization as

$$e(T)=\prod_{q\geq 2} q^{v_q}$$

where v_q= ordinal of $e(T)$ at q. For this prime factorization of $e(T)$, we let $E_e(T)=\{q:v_q\neq 0\}$ and $h_e(T)=\Sigma v_q$. We also make the following definition.

Definition AI-1: The $e^2(T)$ factorization of $D(T)$ is the prime factorization of $D(T)$ as the following

$$D(T)=4^{\ell}\prod_{q\geq 2} q^{2w_q}\prod_{q\geq 2} q^{u_q}$$

where

$$\ell = \begin{cases} 0, & 4\nmid D(T) \\ 1, & 4 \mid D(T), \end{cases}$$

$$w_q \le v_q,$$

and $w_q = v_q$, if $u_q \ge 2$. Here v_q is the ordinal of $e(T)$ at q.

Letting $h_D(T) = \Sigma w_q$, we see that $h_D(T) \le h_e(T)$ since $w_q = 0$ for every $q \notin E_e(T)$.

Now let $p \in E_e(T)$ and recall that $a_w(T)$ is invariant under unimodular transformations, $T \to T[U]$. Then, every $T = (t_1, t_2, t_3) \in \mathfrak{T}$ can be normalized such that $(t_1 e(T)^{-1}, p) = 1$ and $t_3 \ge 0$. To see this, let $t_i = m_i e(T)$. The second condition can always be met if $t_3 < 0$ by $u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}: (t_1, t_2, t_3) \to (t_1, t_2, -t_3)$. If $(m_1, p) > 1$ and $(m_2, p) = 1$, then $u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}: (t_1, t_2, -t_3) \to (t_2, t_1, t_3)$ works. Similarly, if $(m_1, p) > 1$ and $(m_2, p) > 1$, $u = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}: (t_1, t_2, t_3) \to (t_1 + t_2 + t_3, t_2, 2t_1 + t_3)$ works.

<u>Proposition</u> <u>A2</u>(<u>Imprimitive</u> <u>Matrix</u>): For $w \ge 2$ and $T(>0)$ imprimitive, equation (AI-3) is

a recursive equation for the imprimitive matrix coefficients $a_w(T)$ in terms of the primitive matrix coefficients.

Proof: In equation (AI-3) we may assume that the imprimitive matrix $T=(t_1,t_2,t_3)\in\mathfrak{T}$ is normalized as above. The proof will then follow by induction on the number of prime elements composing $e(T)$, namely $h_e(T)$. First of all for a fixed imprimitive matrix $T\in\mathfrak{T}$, we examine the types of matrice S, that will occur on the right hand side of equation (AI-3). Here S is of the form $S=(s_1,s_2,s_3)\in\mathfrak{T}$. Fixing $p\in E_e(T)$ and defining $e=e(T)$ and $D=D(T)$, we have the following results.

(i) If $S=\frac{T}{p}\in\mathfrak{T}$, then $e(S)=p^{-1}e(T)$ and $D(S)=|2S|e^{-2}(S)=|2T|e^{-2}(T)=D(T)$.

(ii) If $S=\frac{T}{p^2}\in\mathfrak{T}$, then the same arguments yield $e(S)=p^{-2}e(T)$ and $D(S)=D(T)$.

(iii) If $S=\frac{1}{p^2}T[\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}]\in\mathfrak{T}$, then $s_1=p^{-2}t_1$, $s_2=t_2$, and $s_3=p^{-1}t_3$. So that, $e(S)=p^{-2}e(T)$ $D(S)=|2S|e^2(S)=p^2D(T)$.

(iv) If $0\le u\le p-1$, $t_i=m_ie$, and $S=\frac{1}{p^2}T[\begin{smallmatrix}u & p\\ 1 & 0\end{smallmatrix}]\in\mathfrak{T}$, then $s_1=p^{-2}(m_1u^2+m_3u+m_2)e$, $s_2=p^{-1}(2m_1u+m_3)e$, and $s_3=m_1e$; so that $|2S|=p^{-2}|2T|$. We then have three cases.

A. $e(S)=p^{-1}e(T)$, $D(S)=D(T)$ iff $p \mid (m_1a^2+m_3u+m_2)$
and $[p^2 \nmid (m_1u^2+m_3u+m_2)$ or $p \nmid (2m_1u+m_3)]$.

B. $e(S)=p^{-2}e(T)$, $D(S)=p^2D(T)$ iff $p \nmid (m_1u^2+m_3u+m_2)$.

C. $e(S)=e(T)$, $D(S)=p^{-2}D(T)$ iff $p^2 \mid (m_1u^2+m_3u+m_2)$
and $p \mid (2m_1u+m_3)$.

With the cases (i-iv) enumerated for a fixed $T\in\mathfrak{J}$ and $p\in E_e(T)$, define $A(T,p)=\{S\in\mathfrak{J}$: S occurs on the right hand side of (AI-3)$\}$. Also define $A(T;e',D')=A(T,p)\cap\{S\in\mathfrak{J}:e(S)=e'$ and $D(S)=D'\}$. In particular, we see that

$$A(T,p)=A(T;p^{-1}e,D)\cup A(T;p^{-2}e,D)$$
$$\cup A(T;p^{-2}e,p^2D)\cup A(T;e,p^{-2}D)$$
$$= A_1 \cup A_2 \cup A_3 \cup A_4 ,$$

so that (AI-3) may be written as

$$\text{(AI-3')} \qquad a_w(T)= \sum_{i=1}^{4} \sum_{S\in A_i} m_1(S)a_w(S),$$

where the $m_i(S)\in \mathbb{Z}$ The convention of equation (AI-3) is also interpreted to mean that in the sets $A_i=A(T;e_i,D_i)$, e_i and D_i must be integers; otherwise $A_i=\emptyset$.

With these statements as preliminaries, we begin the induction proof. Suppose T is an arbitrary (normalized) imprimitive matrix in $\mathfrak{J}$ such that $h_e(T)=1$ and let $p\in E_e(T)$. Then $h_D(T)=0$ or 1 (since $h_D(T)\le h_e(T)$). $h_D(e)=0$ implies A_1 contains only primitive matrices and $A_i=\phi, 2\le i\le 4$. Hence the proposition holds. Similarly, $h_D(T)=1$ implies A_1 contains only primitive matrices, $A_2=A_3=\phi$, and $A_4=\phi$ or $A_4\ne\phi$. If $A_4=\phi$, the proposition holds as before. If $A_4\ne\phi$, then every $S\in A_4$ can be normalized and substituted for T in equation (AI-3) determining the sets $A_i', 1\le i\le 4$. Since $h_D(S)=h_D(T)-1=0$, the arguments above applied to the A_i' determine that $a_w(S)$ for $S\in A_4$ are functions only of primitive matrix coefficients. Hence we conclude that this proposition is true for $T\in\mathfrak{J}$ and $h_e(T)=1$.

Now suppose that the proposition holds for every imprimitive $T\in\mathfrak{J}$ with $h_e(T)=n$. We will show that the proposition holds for every $T\in\mathfrak{J}$ with $h_e(T)=n+1$. To this end, let T (normalized)$\in\mathfrak{J}$ such that $h_e(T)=n+1$. Since the terms of (AI-3') with $S\in A_1, A_2$, and A_3 all have $h_e(S)\le n$, we can write them in terms of primitive matrix coefficients by the inductive hypothesis. As before $A_4=\phi$ or $A_4\neq\phi$. If $A_4=\phi$, we are through. If $A_4\neq\phi$, every $S_1\in A_4$ may be normalized and substituted for T in equation (AI-3) determining the sets A_i', $1\leq i\leq 4$. Again the sets A_i', $1\leq i\leq 3$ cause no problem by the inductive hypothesis and $A_4'\neq\phi$ or $A_4'=\phi$. If $A_4'=\phi$, we are through. If $A_4'\neq\phi$, we repeat the process again for $S_2\in A_4'$. The process will terminate since at each stage $h_e(S_i)=n+1$, but $h_D(S_i)\leq h_D(T)-i$. Thus the proof is complete.

QED

We note that equations (AI-2) and (AI-3) give a method for generating all the coefficients of degree two Eisenstein series. Equation (AI-2) solves the problem for all primitive matrices $T\in\mathfrak{J}$ and equation (AI-3) reduces imprimitive matrix calculations down to primitive matrix calculations.

4. No effort has been made to ultilize the results above for computations of the coefficients in this paper. However, the following congruence relationship for the Eisenstein coefficients of degree 2 and weight $w\geq 4$ has been used as a check in Chapter IV.

$$\text{(AI-4)} \qquad a_w(T) \equiv 0 \left(\bmod \frac{4w(w-1)}{B_w B_{2w-2}}\right)$$

where B_w are the Bernoulli numbers (Chapter I-equation (I-13)). Equation (AI-4) holds for all $T>0$ where d the discriminant is not equal to -4 or $-p$, a prime number. Reference for (AI-4) is Maass' paper [6].

Appendix II: Computer Time and Cost Requirements

The recursive equations for the ϕ_4 and χ_{10} Fourier coefficients (Chapter III - equations III-3 and III-4) were mechanized on an available Control Data Corporation 3800 computer. This task was divided into three distinct programming and verification phases with a fourth phase added for actual computations. These are described as follows:

(1) Mechanization of the algorithm for determining the reduced matrix of an arbitrary matrix $T \in \mathfrak{T}$ (Chapter I-section F).

(2) Mechanization of a method for determining all decompositions (in the form of equation IV-4) for the augmented matrix of $T \in \mathfrak{T}$ (Chapter IV-section B).

(3) Mechanization of the recursive equations for the ϕ_4 and χ_{10} Fourier coefficients given by Propositions (7) and (8). (The method employed here was outlined in Chapter IV-section D and used the results of (1) and (2) above.)

(4) Computation of the ϕ_4 and χ_{10} Fourier coefficients given in Table I (Chapter V).

Since all of these phases were performed on a time permitting and computer availability schedule over a three month period, we have found it extremely difficult to estimate the total time and cost required to perform the entire task. However, for the ϕ_4 and χ_{10} coefficients actually tabulated in Table I (Chapter V), a very accurate account was kept and showed that 17.3 hours of computer time at \$550/hr. (total = \$9,515) was needed. This, of course, does not give a total account for phase (4), since there were numerous runs aborted for one reason or another (computer priorities, computer tape drive failure, etc.). Also, it must be mentioned that, with the exception of upgrading the method used in (2), very little effort was spent in optimizing the computer program.

Table AII-1 gives a breakdown of the time mentioned above. Here the computational time needed for each ϕ_4 and χ_{10} coefficient is tabulated versus the matrix of the coefficient. Since it was found that the computations in (2) required the most time, it is obvious from the table why we had to revamp the method used for determining the decompositions. We do not look upon this change as a big sophistication of the program; however, with the change a substantial decrease in time required was achieved. This change gave a more predictable rate of growth to the time required.

To conclude this appendix, we have tabulated in Table AII-2 the number of decompositions, (C_T), of a matrix T used in the recursive equations (III-3) and (III-4). From these values, it is seen that the growth rate of time required (after the change above) is very similar to the growth rate of the number of terms in the recursive equations used. This, of course, is as it should be.

Table AII-1

Computation Time for the ϕ_4 and χ_{10} Coefficients

T	Computation time-minutes*	T	Computation time-minutes*
(1,2,1)	2.1	(3,3,3)	9.9
(1,2,0)	1.2	(3,3,2)	11.7
(1,3,1)	6.4	(3,3,1)	14.7
(1,3,0)	2.3	(3,3,0)	17.4
(2,2,2)	23.1	(1,6,1)	4.8
(2,2,1)	11.7	(1,6,0)	6.4
(2,2,0)	3.5	(2,5,2)	19.8
(1,4,1)	15.0	(2,5,1)	24.3
(1,4,0)	4.9	(2,5,0)	28.8
(2,3,2)	68.4	(3,4,3)	24.4
(2,3,1)	24.9	(3,4,2)	30.6
(2,3,0)	10.8	(3,4,1)	38.6
**(1,5,1)	2.2	(3,4,0)	58.3
(1,5,0)	3.6	(1,7,1)	9.3
(2,4,2)	4.2	(1,7,0)	11.2
(2,4,1)	7.0	(2,6,2)	34.5
(2,4,0)	10.3	(2,6,1)	39.7

* Sum of the computation time for the ϕ_4 and χ_{10} coefficients.

** Reprogrammed at this point.

Table AII-1 (continued)

T	Computation time-minutes
(2,6,0)	44.8
(3,5,3)	74.1
(3,5,2)	81.7
(3,5,1)	98.3
(3,5,0)	117.3
(4,4,4)	119.7
(1,8,1)	12.5
(1,8,0)	14.7
(2,7,1)	83.8
(1,9,1)	28.9
(1,9,0)	32.7

Table AII-2

Number of Terms in the Recursive Equations (III-3) and (III-4)

T	Left Hand Side of Equation III-4	Right Hand Side of Equation III-4	Right Hand Side of Equation III-3
(1,2, 1)	16	17	4
(1,2, 0)	40	44	6
(1,3, 1)	100	117	6
(1,3, 0)	176	220	8
(2,2, 2)	64	77	9
(2,2, 1)	160	207	10
(2,2, 0)	284	394	13
(1,4, 1)	332	449	8
(1,4, 0)	516	736	10
(2,3, 2)	400	573	14
(2,3, 1)	728	1,147	16
(2,3, 0)	1,112	1,863	20
(1,5, 1)	853	1,302	10
(1,5, 0)	1,212	1,948	12
(2,4, 2)	1,384	2,429	21
(2,4, 1)	2,212	4,187	24
(2,4, 0)	3,072	6,172	27

Table AII-2- (Continued)

T	Left Hand Side of Equation III-4	Right Hand Side of Equation III-4	Right Hand Side of Equation III-3
(3,3,3)	1,000	1,635	22
(3,3,2)	1,856	3,411	26
(3,3,1)	2,924	5,826	28
(3,3,0)	4,016	8,534	32
(1,6,1)	1,840	3,142	12
(1,6,0)	2,480	4,428	14
(2,5,2)	3,676	7,637	30
(2,5,1)	5,344	11,939	32
(2,5,0)	7,032	16,528	34
(3,4,3)	3,556	7,443	34
(3,4,2)	5,781	13,313	40
(3,4,1)	8,272	20,538	42
(3,4,0)	10,764	28,184	44
(1,7,1)	3,542	6,684	14
(1,7,0)	4,596	9,024	16
(2,6,2)	8,128	19,734	39
(2,6,1)	11,168	28,871	40
(2,6,0)	14,112	38,215	43
(3,5,3)	9,694	24,873	50

Table AII-2- (Continued)

T	Left Hand Side of Equation III-4	Right Hand Side of Equation III-4	Right Hand Side of Equation III-3
(3,5,2)	14,320	40,142	56
(3,5,1)	--	--	-
(3,5,0)	--	--	-
(4,4,4)	6,920	16,763	47
(1,8,1)	6,280	12,964	16
(1,8,0)	7,896	16,920	18
(2,7,1)	15,976	44,494	48
(1,9,1)	10,435	23,399	18
(1,9,0)	12,800	29,720	20

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