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Degeneration of minimal surfaces in the bidisc

By

Charles Ouyang

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APPROVED, THESIS COMMITTEE

Michael Wolf
Michael Wolf (Apr 21, 2020)

Michael Wolf
Professor of Mathematics

Robert M. Hardt
Robert M. Hardt (Apr 21, 2020)

Robert Hardt
W.L. Moody Professor of Mathematics

Andriy Nevidomskyy
Associate Professor of Physics and Astronomy

HOUSTON, TEXAS
April 2020

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Abstract

This thesis studies the degeneration of a particular class of minimal surfaces in the bidisc, describing both the limiting metric structure and geometry. Minimal surfaces inside symmetric spaces have been shown to be directly related to surface group representations into higher rank Lie groups by recent work of Labourie. Let S be a closed surface of genus $g \geq 2$ and let ρ be a maximal $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ surface group representation. By a result of Schoen, there is a unique ρ-equivariant minimal surface $\widetilde{\Sigma}$ in $\mathbb{H}^2 \times \mathbb{H}^2$. We study the induced metrics on these minimal surfaces and prove the boundary limits are precisely mixed structures, as defined below in the introduction. In the second half of the thesis, we provide a geometric interpretation: the minimal surfaces $\widetilde{\Sigma}$ degenerate to the core of a product of two $\mathbb{R}$-trees. As a consequence, we obtain a geometric compactification of the space of maximal representations of $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

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Chapter 1

Introduction and background

1.1 A lay introduction

Since the dawn of civilization, man has gazed up towards the heavens and wondered what lied beyond. Explorers from empires past have voyaged to the ends of the earth to chart new lands. But the quest to understand what things "far away" look or how things change also appears in mathematics. Archimedes looked at regular n-sided polygons, and for large n, observed the shape looked more and more circular. Centuries later, with the advent of calculus, limits of functions were studied.

The thesis seeks to describe particular limits. But instead of celestial bodies or behavior of functions, the objects we are interested in are minimal surfaces. A *minimal surface* is a surface that locally minimizes area. In Euclidean 3-space $\mathbb{R}^3$, physical models of minimal surfaces can be seen with soap solutions adhering to some wire frame creating a soap film. These configurations are nice models of minimal surfaces, as nature seeks to minimize "energy", and any configuration occupying the least area would be a prime candidate. The particular class of minimal surfaces we are interested in however, cannot be easily physically modeled, as is in the case of minimal surfaces in $\mathbb{R}^3$. We are mainly interested in minimal surfaces in the bidisc $\mathbb{H}^2 \times \mathbb{H}^2$, which is a 4-dimensional space. The hyperbolic space $\mathbb{H}^2$ is the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ endowed with the hyperbolic metric $ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$. One possible way to envision this metric is the disk is covered with a strange liquid, and if one starts at the origin of the disk and wades further out, one can imagine the liquid become more and more viscous. Hence, what we might usually perceive as a small distance in the Euclidean sense, becomes larger and larger, the further out from the center of the disk we are. The bidisc is then the product of two copies of $\mathbb{H}^2$. Our choice of studying minimal surfaces in $\mathbb{H}^2 \times \mathbb{H}^2$ is not arbitrary, but rather because it appears as the symmetric space of the Lie group $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

In what may initially appear as an entirely different focus, we are interested in surface group representations to $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. A *surface group* is the fundamental group $\pi_1(S)$ of a closed surface S. It is the group of (equivalence classes under homotopy) of closed loops on the surface S. Concretely, for a surface of genus g, it has the following group presentation

$$\pi_1(S) = \{\gamma_1, \gamma_2, ..., \gamma_{2g-1}, \gamma_{2g} : \prod_{i=1}^{g} \gamma_{2i-1}\gamma_{2i}\gamma_{2i-1}^{-1}\gamma_{2i}^{-1} = 1\}.$$

By a *representation*, we mean a homomorphism between groups, so that the surface group representations to the Lie group $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ will be a group homomorphism between $\pi_1(S)$ and $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

It may be clear at this point why will restrict ourselves to the case where the genus g of the surface S is at least 2. In the genus 0 case, the fundamental group is trivial, as the sphere is simply connected. In the genus 1 case, though the fundamental group is nontrivial, it is abelian, and from a representation point of view, becomes too simplistic.

In order to motivate why these algebraic objects are particularly interesting, we use as a model case of surface group representations to $\mathrm{PSL}(2,\mathbb{R})$. This is where classical Teichmüller theory comes into play. As the intent of this section is to provide an introduction to a non-specialist audience, we defer precise definitions to the next section.

The group $\mathrm{PSL}(2,\mathbb{R})$ is defined by

$$\mathrm{PSL}(2,\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \text{ and } ad - bc = 1 \right\} / \{\pm I\},$$

so it is the quotient of the group $\mathrm{SL}(2,\mathbb{R})$ by identifying the identity element I with its negative $-I$. The group $\mathrm{PSL}(2,\mathbb{R})$ is the group of isometries of $\mathbb{H}^2$. If we have a representation ρ from $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R})$, that is both faithful (injective) and discrete, then the quotient $\mathbb{H}^2/\rho(\pi_1(S))$ is a closed hyperbolic surface with the same underlying topology as S. These particular surface group representations into $\mathrm{PSL}(2,\mathbb{R})$, which are both discrete and faithful, are known as *Fuchsian* representations. In fact, if one starts with a closed hyperbolic surface X with the same underlying smooth surface S, one can lift to its universal cover $\widetilde{X}$, which by the Cartan-Hadamard theorem, is isometrically diffeomorphic to $\mathbb{H}^2$. Under the usual correspondence between elements of the fundamental group of X to deck transformations of $\widetilde{X}$, one then obtains from a closed hyperbolic surface X, a Fuchsian representation.

Herein we see a bridge between two different fields of mathematics. On the one hand we have a surface group representation to $\mathrm{PSL}(2,\mathbb{R})$, which is an algebraic and topological object. The algebraic aspect is clear, as we are dealing with a group homomorphism. The topological feature comes from the fundamental group (the fundamental group of a closed surface completely determines the surface up to homeomorphism). On the other hand, we have a hyperbolic surface, a geometric object. The metric allows us to makes sense of lengths and angles.

One final class of objects, we would like to introduce, are Riemann surfaces. Informally (the precise definition is given in the next section), these are patches of the complex plane $\mathbb{C}$ stitched together in a holomorphic (complex differentiable) fashion. Though Riemann surfaces will naturally carry an underlying smooth structure S, for a fixed surface S, there are many different possible Riemann surface structures one may place on the same surface S. The *moduli space* of Riemann surfaces is the space of possible Riemann surface (or complex) structures one may place on S. Riemann himself was interested in this particular question of moduli, and knew the number of parameters on which the complex structure depends. Classical Teichmüller theory is the study of Riemann surfaces and their variations.

A Riemann surface is a holomorphic object, and having the notion of locally resembling the complex plane, allows us to make sense of angles (given locally by multiplication by i). Notice that having the notion of angle is not quite enough if we desire to work with a metric. We would need the notion of length. But the

Uniformization theorem allows one to start with just notions of angle and extend this to an apt metric, which is usually taken to be the hyperbolic metric, thus allowing one to freely pass between Riemann surfaces and hyperbolic surfaces.

The Uniformization theorem completes the bridge between holomorphic objects and geometric ones. Combining with the earlier discussion concerning Fuchsian representations and hyperbolic surfaces, we now see a correspondence between algebraic/topological objects, geometric objects and holomorphic objects.

It is one aim of higher Teichmüller theory to extend such a correspondence to include different groups, not just $\mathrm{PSL}(2,\mathbb{R})$. In replacing $\mathrm{PSL}(2,\mathbb{R})$ with a higher rank Lie group G, the resulting algebraic/topological objects are still surface group representations, though this time not to $\mathrm{PSL}(2,\mathbb{R})$, but rather to G. Hyperbolic metrics are replaced with minimal surfaces inside a symmetric space and are the higher rank versions of the geometric objects. Finally, Riemann surfaces will be replaced with Riemann surfaces with a holomorphic n-differential.

One possible question is why in particular choose $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$? In the later sections on background material, we will see that the following groups enjoy the correspondence discussed above: $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{Sp}(4,\mathbb{R})$ and G_2. In higher rank, the correspondence is conjectured to be true, but remains unproven. Among the rank 2 groups, $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ has a product structure allowing us to employ some techniques coming from the theory of representations to $\mathrm{PSL}(2,\mathbb{R})$. These tools cannot immediately be applied to the other groups, though in the concluding remarks, we will mention how some of the methods developed in this thesis can be used to study the other rank 2 groups (see [49], [50]).

Our thesis studies the group $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. Utilizing the correspondence discussed above, observing that the symmetric space associated to $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ is the bidisc $\mathbb{H}^2 \times \mathbb{H}^2$, it is clear now why minimal surfaces in the bidisc have interest outside of geometry. These minimal surfaces correspond to surface group representations to $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. For sake of completeness, we remark the analogous holomorphic objects are Riemann surfaces with a holomorphic quadratic differential, which will play a prominent role in the setting of harmonic maps later in our work. One can now see from this correspondence, why if one had interests in surface group representations $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ and their limits, why it is worthwhile to study the behavior of minimal surfaces in $\mathbb{H}^2 \times \mathbb{H}^2$ as one changes the representation.

In our title, we have opted for the word degeneration instead of limits. This is to emphasize that the minimal surfaces we study, start to change into a new class of objects of lower complexity. Whereas minimal surfaces are defined by solutions to a partial differential equation, the cores of $\mathbb{R}$-trees (to be defined in Chapter 3), are defined by prescribing an explicit recipe to remove pieces of a space. What remains is called the core.

This thesis contains four chapters. The first contains introductory remarks and provides the initial geometric framework for the thesis. We provide a cursory review of classical Teichmüller theory and some of the topological and analytic tools used, namely geodesic currents and harmonic maps. We then discuss the basics of the higher rank Teichmüller theory, which has been a relatively new and popular, developing field of geometry. The second chapter contains our first main result. Here we study the variational quantities related to harmonic maps in order to make a statement about minimal surfaces. This allows us to proceed with our goal of compactifying a particular class of metrics on these minimal surfaces. Chapter three is the bulk of the thesis and contains the majority of our new results. We classify all the boundary limits of the minimal surfaces in the bidisc and describe the limiting metric structure and geometry. Finally we end the thesis with chapter four which consists of future projects and related questions.

1.2 A non-lay introduction

Classical Teichmüller theory studies the space of marked hyperbolic structures or equivalently the space of marked Riemann surfaces. (Recall a *marking* is a fixed labelling of elements of the fundamental group of a surface S.) But one may also view the Teichmüller space as the representation variety of conjugacy classes of discrete and faithful surface group homomorphisms into $\mathrm{PSL}(2, \mathbb{R})$.

When the Lie group is of higher rank, surface group representations do not immediately correspond to geometric objects. In particular low rank settings and when the representation is Hitchin, recent work by Labourie [38] shows to each such representation ρ, there is a unique ρ-equivariant minimal surface in the associated symmetric space.

It is from this perspective we wish to conduct our study of limits of representations. The first of these lower high rank settings is the semisimple Lie group $G = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$, which is studied extensively in this thesis. Here, the symmetric space is

$\mathbb{H}^2 \times \mathbb{H}^2$. Using the disc model of hyperbolic 2-space $\mathbb{H}^2$, then one sees the symmetric space $\mathbb{H}^2 \times \mathbb{H}^2$ is the bidisc $\mathbb{D} \times \mathbb{D}$ equipped with the product metric, where each factor has the usual Poincaré metric. The Hitchin representations into $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ correspond to minimal lagrangians in the bidisc, which are parameterized by two copies of Teichmüller space.

There is a goal in the higher Teichmüller theory of understanding limits of representations as geometric objects generalizing measured laminations, which occur as boundary limits in classical Teichmüller theory. One wants to see the sequences of representations leaving all compact sets somehow as geometric objects which degenerate to the objects at the boundary, in the same spirit as the Thurston compactification (see section 11 of [60]). This thesis addresses this goal in the case of $G = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. The intermediate geometric objects are minimal lagrangians in the bidisc, and the objects at the boundary are cores in the sense of Guirardel (see [28]) of a product of two $\mathbb{R}$-trees constructed from the data of a projective pair of measured laminations. We will show that this interpretation of cores will coincide with the Thurston compactification when we restrict our compactifcation to classical Teichmüller space.

The first part of our main results studies the metric structure of the minimal langrangians using marked length spectra and geodesic currents. This perspective was first utilized by Bonahon [3] in the setting of Teichmüller space, and he recovers the Thurston compactification in this way.

We study the length spectrum of these induced metrics on the minimal surface and show that we can degenerate the metrics to obtain singular flat metrics, measured laminations and mixed structures. A *mixed structure* $\eta = (S_\alpha, q_\alpha, \lambda)$ is the data of a collection of incompressible subsurfaces S_α, with a prescribed meromorphic (integrable) quadratic differential on each subsurface (collapsing the boundary components and viewing them as punctures) and a singular flat metric on each subsurface coming from the prescribed quadratic differential, with a measured lamination λ supported on the complement $S \setminus \sqcup S_\alpha$. Observe that a singular flat metric coming from a holomorphic quadratic differential on (S, J) and a measured lamination on S are trivial examples of mixed structures, corresponding to $S_\alpha = S$ and $S_\alpha = \emptyset$, respectively. Our first main result is the following.

Theorem 1.1. *The space of induced metrics* $\mathrm{Ind}(S)$ *embeds into the space of projectivized currents* $\mathrm{PCurr}(S)$. *Its closure is* $\mathrm{Ind}(S) \sqcup \mathrm{PMix}(S)$, *where* $\mathrm{PMix}(S)$ *is the space of projectivized mixed structures on* S *.*

If we keep track of the ambient space, namely $\mathbb{H}^2 \times \mathbb{H}^2$, we show that by scaling the ambient space by a suitable sequence of constants (which generally will be the total energy of some harmonic maps), we can obtain, as limits of minimal langrangians, the core of a pair of $\mathbb{R}$-trees coming from measured foliations. These will be distinguished subsets of a product of trees, which will be typically part 2-dimensional, and part 1-dimensional. These objects will be defined in a geometric group-theoretic way, requiring only the data of the pair of $\mathbb{R}$-trees and the group action. In fact, we show there is an isometry from a metric space obtained from the data of a mixed structure to the core of trees.

As a consequence, we have an answer to our original goal of ascribing something geometric to a surface group representation to $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ which is maximal, and a description of a natural boundary object which is geometric and is a natural extension of measured laminations.

Theorem 1.2. *The space of maximal representations of* $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ *embeds into the space of $\pi_1 S$-equivariant harmonic maps from $\mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2$, whose graphs are minimal lagrangians. The scaled Gromov-Hausdorff limits of these graphs are given by harmonic maps from $\mathbb{H}^2$ to $T_1 \times T_2$, where T_1 and T_2 are a pair of $\mathbb{R}$-trees coming from a projective pair of measured foliations, with image given by the core of the trees.*

There has been some recent interest in studying surface group representations to the Lie group $\mathrm{PSL}(2,\mathbb{R})\times \mathrm{PSL}(2,\mathbb{R})$ by way of geodesic currents. Work of Glorieux [24] shows that the average of two Liouville currents $\frac{L_{X_1}+L_{X_2}}{2}$ yields the length spectrum of the Globally Hyperbolic Maximal Compact AdS^3 manifold with holonomy (ρ_1, ρ_2), where $X_i = \mathbb{H}^2 \setminus \rho_i$. In another recent paper of Glorieux [25], it is shown that this map which sends an unordered pair of elements in Teichmüller space to a projectivized current given by $(X_1, X_2) = (X_2, X_1) \to \frac{L_{X_1}+L_{X_2}}{2}$ is injective. Forthcoming work of Burger, Iozzi, Parreau, and Pozzetti [6] will show the limits of this embedding are given by the projectivization of a pair of measured laminations. Their limiting currents thus satisfy

$$i(\eta, \cdot) = i(\lambda_1, \cdot) + i(\lambda_2, \cdot), \tag{1.1}$$

where λ_1 and λ_2 are specific representatives of the projectivized classes $[\lambda_1]$ and $[\lambda_2]$, respectively, representing limits on the Thurston boundary.

We remark that our compactifcation via geodesic currents is distinct. If the limiting laminations λ_1 and λ_2 *fill*, that is, the sum of their intersection numbers with any third measured lamination is never zero, then the corresponding limiting object η' under our compactification is a singular flat metric coming from a unit-norm holomorphic quadratic differential Φ whose horizontal and vertical laminations are λ_1 and λ_2. The corresponding current is thus given by

$$l^2_{|\Phi|}(\alpha) = i^2(\eta', \alpha) = i^2(\lambda_1, \alpha) + i^2(\lambda_2, \alpha), \tag{1.2}$$

for a suitably short arc α away from the zeros of $|\Phi|$. In general, this is different from the sum of λ_1 and λ_2. Notice that for γ an arc of the horizontal lamination of Φ, then the two intersection numbers $i(\eta, \alpha)$ and $i(\eta', \alpha)$ coincide, so that the two currents η and η' are distinct even as projectivized currents.

One may view the work [6] of Burger, Iozzi, Parreau and Pozzetti as understanding the limiting length spectra of degenerating families of globally hyperbolic maximally Cauchy-compact anti-de Sitter 3-manifolds (henceforth abbreviated as GHMC AdS^3 manifolds; for a more detailed discussion, see Section 3.4), whereas our work in investigating the metric structure of the minimal langrangians in the bidisc furnishes the limiting data of the unique embedded space-like maximal surface in the AdS^3 manifold. This is not the main perspective we adopt, as it is unclear how the AdS^3 manifold converges geometrically, and so one may possibly lose information on the ambient space. By contrast, this is not the case with minimal lagrangians in the bidisc.

1.3 Geometric Preliminaries

We begin by summarizing the underlying geometric objects required throughout the thesis.

1.3.1 Riemann surface theory

A *Riemann surface* X is a one-dimensional complex manifold. It is a smooth surface which admits a maximal atlas of charts into the complex plane $\mathbb{C}$ with transition maps being biholomorphisms. A compact Riemann surface will be a Riemann surface, whose underlying smooth structure is a closed (compact with no boundary) surface. For our purposes, Riemann surfaces will be compact Riemann surfaces unless otherwise indicated. A punctured Riemann surface will be obtained from a closed surface by removing finitely many points. Open neighborhoods containing the punctures will be mapped onto a punctured disc in the complex plane.

The real cotangent bundle of X is denoted T^*X and may be complexified to yield the complex cotangent bundle $T^*X \otimes \mathbb{C}$. The underlying complex structure on X allows us to decompose any smooth covector into a holomorphic $(1,0)$-part and an antiholomorphic $(0,1)$-part. The canonical bundle K_X of X will be the $(1,0)$-part of the complexified cotangent bundle. When the underlying Riemann surface X is clear, the canonical bundle will be simply denoted as K.

Natural operations may be performed on bundles. In particular, the tensor product of two bundles is well-defined, and to any bundle, the vector space of sections is well-defined. A *section* is a smooth map from the surface to the total space of the bundle, so that postcomposition with the projection map yields the identity map.

A *holomorphic quadratic differential* is a holomorphic section of the square of the canonical bundle. The vector space of holomorphic quadratic differentials is denoted $H^0(X, K^2)$. One corollary of the Riemann-Roch theorem is that the complex dimension of $H^0(X, K^2)$ is $3g - 3$, where g is the genus of the underlying smooth surface. More concretely, if X is a Riemann surface and Φ is a holomorphic quadratic differential, then locally $\Phi = f(z)dz^2$, where f is holomorphic and z is a coordinate chart for X.

If $S_{g,n}$ is a compact surface of genus g with n punctures such that $3g-3+n > 0$, then $Q_{g,n}$ will denote the space of integrable holomorphic quadratic differentials on $S_{g,n}$. At each of the punctures, the differential has at worst a pole of order 1.

1.3.2 Classical Teichmüller theory

Teichmüller space is the space of marked hyperbolic structures on a surface S. If X is a hyperbolic surface, a *marking* is a diffeomorphism $\psi : X \to S$. This allows one to keep track of homotopy classes of curves on X. Two hyperbolic metrics g, h are identified if there exists a diffeomorphism ϕ isotopic to the identity map, so that the pullback metric $\phi^* g$ is equal to h. The topology is given by its *marked length spectrum.*

Alternatively, Teichmüller space may be regarded as the space of marked Riemann surface structures on S. Two complex structures X and Y on S are identified, if there is a biholomorphism $\phi : X \to Y$, which is isotopic to the identity map as a smooth map.

The last setting in which Teichmüller space may be viewed, is from the perspective of surface group representations. A *representation* will be a homomorphism from the fundamental group $\pi_1(S)$ of S to a Lie group G. Here, Teichmüller space is the space of discrete and faithful surface group representations into the Lie group $\mathrm{PSL}(2,\mathbb{R})$. Two representations are identified if one may be conjugated to the other.

One may freely pass between the various incarnations of Teichmüller space. The hyperbolic perspective and the Riemann surface viewpoint are equivalent by the Uniformization theorem. From a surface group representation, one constructs an hyperbolic surface by taking the quotient of $\mathbb{H}^2$ by the representation. That is to say, if ρ is a discrete and faithful representation of the fundamental group $\pi_1(S)$ of a surface S into the isometry group $\mathrm{PSL}(2,\mathbb{R})$ of $\mathbb{H}^2$, the quotient $\mathbb{H}^2/\rho(\pi_1(S))$ is a closed surface with the same underlying topology of S, but now inherits a hyperbolic metric obtained from $\mathbb{H}^2$. From a hyperbolic structure, one recovers the representation, by taking its holonomy representation. Closed geodesics on the hyperbolic surface lift to isometries of $\mathbb{H}^2$. Using this, one may construct a Fuchsian representation from the data of a hyperbolic surface.

Teichmüller space is topologically trivial, being homeomorphic to $\mathbb{R}^{6g-6}$.

1.3.3 The Thurston compactification

The Thurston compactification utilizes the perspective of Teichmüller space as the space of marked hyperbolic surfaces. Let $\mathcal{S} = \mathcal{S}(S)$ denote the set of free isotopy classes of simple closed curves on S. Fix a hyperbolic metric m. To any element $[\gamma]$ of $\mathcal{S}$, the length of the m-geodesic of $[\gamma]$ is well-defined and is a positive number. With this map, to each element of Teichmüller space, one obtains a sequence of positive numbers indexed by elements of $\mathcal{S}$, where each number $l_m([\gamma])$ is the length of the m-geodesic. We will refer to this sequence as the *marked length spectrum* of m. Call the map

$$\mathcal{L} : \mathrm{Teich}(S) \to \mathbb{R}^{\mathcal{S}}_{\geq 0}$$

It is a classical result (see [19]) that there are $9g - 9$ simple closed curves, whose lengths determine the hyperbolic metric up to isotopy homotopic to the identity. Hence the map $\mathcal{L}$ above is seen to be injective.

In constructing a compactification of a space X, it is often useful to embed the space X into a compact space Y, then take the closure of X in Y, or to embed the space X into a space Z with precompact image. The latter is done by Thurston [19] with Teichmüller space. Consider the space $\mathbb{PR}^{\mathcal{S}}_{\geq 0} = \mathbb{P}(\mathbb{R}^{\mathcal{S}} - \{0\})$, which is the space $\mathbb{R}^{\mathcal{S}}_{\geq 0}$ with the sequence which is identically 0 removed, up to scalar multiplication. There is a natural projection map $\pi : \mathbb{R}^{\mathcal{S}}_{\geq 0} \to \mathbb{PR}^{\mathcal{S}}_{\geq 0}$. Then using some elementary hyperbolic geometry, it is shown that $\pi \circ \mathcal{L} : \mathrm{Teich}(S) \to \mathbb{PR}^{\mathcal{S}}_{\geq 0}$ is injective.

Theorem 1.3 (Thurston [19])**.** *The map*

$$\pi \circ \mathcal{L} : \mathit{Teich}(S) \to \mathbb{PR}^{\mathcal{S}}_{\geq 0}$$

is injective with precompact image.

The boundary is described explicitly. It is important to remark here that the boundary points are not simply arbitrary non-negative sequences, but rather, are the marked length spectra of a topological object. The boundary points under this compactification are given by projective classes of measured foliations. A *measured foliation* on a closed surface S is a singular foliation with a transverse measure, that is a measure μ defined on each arc transverse to the foliation, such that the measure is invariant under isotopy between two arcs through transverse arcs. Measured foliations are considered equivalent if they differ by an isotopy or *Whitehead equivalence*, which consists of collapsing arcs between singularities.

Thurston shows the space $\mathcal{MF}(S)$ of measured foliations on a surface S is homeomorphic to $\mathbb{R}^{6g-6}$, so that $\mathbb{P}\mathcal{MF}(S)$ is homeomorphic to a sphere of dimension $6g-7$. The boundary of the Thurston compactification is thus a sphere. A particularly salient feature of this compactification is that the boundary is acted upon naturally by the mapping class group.

It is appropriate to mention that there is a natural relation between measured foliations and holomorphic quadratic differentials. Holomorphicity of the differential and compactness of the Riemann surface ensures the quadratic differential has precisely $4g-4$ zeros counted with multiplicity. Hence, in a simply-connected neighborhood avoiding a zero of Φ, one may choose natural coordinates ζ so that $\Phi = d\zeta^2$. The metric $|\Phi|$ is well-defined on the complement of the zeros and is locally Euclidean. At the zeros, the metric has conic singularities of angle $(n+2)\pi$, where n is the order of the zero of the quadratic differential at that point.

For any point in the complement of the zeros of the quadratic differential, there is a unique direction for which $q(v,v) \in \mathbb{R}^+$. Integrating the resulting line field, one obtains a foliation, called the *horizontal foliation* of the quadratic differential q.

Likewise, one can define the *vertical foliation* of q, by integrating the line field of directions for which $q(v, v) \in i\mathbb{R}^+$. The foliations come equipped with a transverse measure. For any arc γ transverse to the horizontal foliation, the measure for the horizontal foliation is given by

$$\tau_h = \int_\gamma |\mathrm{Im}(\sqrt{q})(z)||dz|,$$

and likewise, the transverse measure for the vertical foliation is given by integrating the real part $|\mathrm{Re}(\sqrt{q})|$ over and arc γ.

1.3.4 Geodesic Currents

Let (S, σ) be a fixed closed hyperbolic surface of genus $g \geq 2$. Then its universal cover $\tilde{S}$ may be identified isometrically with $\mathbb{H}^2$. Let $G(\tilde{S})$ denote the space of geodesics of $\tilde{S}$. Then a *geodesic current* on S is a $\pi_1(S)$-equivariant Radon measure on $G(\tilde{S})$. The space of geodesic currents, denoted Curr(S), is given by the weak* topology.

Remark 1.4. A priori, the definition of a geodesic current may appear to depend upon the choice of hyperbolic metric, but it turns out $G(\tilde{S})$ depends only upon $\pi_1(S)$ (c.f. [3]), hence the space of geodesic currents is independent of the hyperbolic metric initially chosen for S.

The ur-example of a geodesic current is given by a single closed geodesic γ on S. Lift γ to a discrete set of geodesics $\tilde{\gamma}$ on $\tilde{S}$. These lifted geodesics may be given a Dirac-measure, which is $\pi_1(S)$-invariant as the lifts themselves are $\pi_1(S)$-invariant. Hence to any closed curve, by looking at its geodesic representative, one obtains a geodesic current on S. In fact, Bonahon [3] shows the space of weighted closed curves is dense in $\mathrm{Curr}(S)$ and the geometric intersection number between curves has a continuous bilinear extension to $i : \mathrm{Curr}(S) \times \mathrm{Curr}(S) \to \mathbb{R}_{\geq 0}$. Moreover, a geodesic current on S is determined by its intersection number with all closed curves [47]. The topology then on the space of geodesic currents is given by its marked length spectrum. In particular, for the fixed surface S, denote by $\mathcal{C}(S)$ the set of isotopy classes of closed curves of S. The marked length spectrum of a geodesic current μ is given by the collection $\{i(\mu, \gamma)\}_{\gamma \in \mathcal{C}(S)}$. We make two remarks. First, the use of the phrase *marked length spectrum* in the context of geodesic currents will be a generalization with that of the marked length spectrum of a hyperbolic surface, as to hyperbolic metric m, there is an associated Liouville current L_m (see [3] for the explicit construction) so that

$$l_m([\gamma]) = i(L_m, \gamma).$$

Hence, the m-length of the geodesic in the homotopy class of γ is equal to the intersection of the currents L_m and γ (here γ is a geodesic current, as constructed above). The second remark is that with geodesic currents, if our indexing set for the marked length spectrum is simply $\mathcal{S}$, then Otal [47] has shown it is not sufficient to distinguish different geodesic currents. However, if one expands the indexing set to be all closed curves on S, then it is sufficient to distinguish geodesic currents based on their marked length spectra.

A sequence of geodesic currents μ_n is said to converge to μ if its marked length spectrums converge, that is, to each $\gamma \in \mathcal{C}(S)$ and $\epsilon > 0$, there is an $N(\epsilon, \gamma)$ so that for $n > N(\epsilon, \gamma)$, one has $|i(\mu, \gamma) - i(\mu_n, \gamma)| < \epsilon$. It is important to note that N is allowed to depend on the curve class chosen. No requirement on uniform convergence is required.

If a current arises from a metric, the following rather useful formula applies:

Proposition 1.5 (Bonahon [3], Otal [47]). *Let μ be a current arising from a metric σ. Then*

$$i(\mu, \mu) = \frac{\pi}{2} Area(\sigma)$$

In the case where μ is a geodesic current arising from a measured lamination, it is not hard to see that $i(\mu, \mu) = 0$, but in fact, this turns out to be a characterization of measured laminations.

Proposition 1.6 (Bonahon [3]). *Let μ be a geodesic current such that $i(\mu, \mu) = 0$, then μ is a measured lamination.*

It is clear that if μ is a geodesic current, then so is $c\mu$ for $c \in \mathbb{R}_+$. The set of projectivized currents, denoted $\mathrm{PCurr}(S)$ is given by $\mathrm{Curr}(S)/\sim$, where $\mu \sim \nu$ if there exists a positive constant c for which $\mu = c\nu$ and so consists of projective classes of geodesic currents. The space $\mathrm{PCurr}(S)$ is then given the quotient topology. We highlight an important property of this space.

Proposition 1.7 (Bonahon [3]). *The space* $\mathrm{PCurr}(S)$ *is compact.*

Several geometric structures have been shown to be embedded into $\mathrm{Curr}(S)$. The first such example was due to Bonahon [3], who showed Teichmüller space could be embedded inside $\mathrm{Curr}(S)$ via its Liouville current, namely $\sigma \mapsto L_\sigma$ with the property that for any closed curve γ, one has $l_\sigma([\gamma]) = i(L_\sigma, \gamma)$, so that the length of the geodesic representative of γ with respect to the hyperbolic metric σ coincides with the intersection number between the currents L_σ and γ. As the space of measured laminations can be realized as geodesic currents, Bonahon recovers the Thurston compactification by way of projectivized geodesic currents.

Otal [47] has shown the space of negatively curved Riemannian metrics on surfaces can be realized by geodesic currents. For any simple curve class $[\gamma]$, the length of the unique geodesic representative coincides with the intersection number of the corresponding geodesic current and the curve class $[\gamma]$, extending the work of Bonahon.

Duchin, Leininger and Rafi [16] have embedded the space of singular flat metrics arising from integrable holomorphic quadratic differentials into the space of geodesic currents. We summarize a few of results here, as we shall use them in what follows. Recall that to any holomorphic quadratic differential q, one can associate a singular flat metric $|q|$ via canonical coordinates.

The unit sphere $Q^1_g \subset Q_g$ consists of the holomorphic quadratic differentials with L^1-norm 1. Then the space $\mathrm{Flat}(S)$ of unit-norm singular flat metrics may be identified by

$$\mathrm{Flat}(S) = Q^1_g \,/\, \mathbb{S}^1,$$

where the action of $\mathbb{S}^1$ is given by multiplication by $e^{i\theta}$, for $0 \leq \theta \leq 2\pi$. We require this quotient because if q is a holomorphic quadratic differential, then q and $e^{i\theta}q$ will have the same singular flat metric $|q|$. For $q \in Q^1_g$, consider the vertical foliation of q, that is $v_q = |\mathrm{Re}(\sqrt{q})|$. Denote $v^\theta_q = |\mathrm{Re}(e^{i\theta}\sqrt{q})|$, the vertical foliation of $e^{i\theta}q$. Form the integral

$$L_q := \frac{1}{2}\int_0^{\pi} v_q^{\theta}\, d\theta.$$

Theorem 1.8 (Duchin-Leininger-Rafi [16])**.** *The integral L_q is a geodesic current such that to any simple closed curve γ,*

$$l_{|q|}(\gamma) = i(L_q, \gamma),$$

where $|q|$ is the singular flat metric arising from the holomorphic quadratic differential q. Furthermore, the map which sends $|q| \in \mathrm{Flat}(S)$ to $L_q \in \mathrm{PCurr}(S)$ is an embedding.

As the space of projectivized currents is compact, one may take the closure of the space $\mathrm{Flat}(S)$, and it is shown [16] that the limiting structures consist precisely of projectivized mixed structures. A *mixed structure* may be defined as follows.

Definition 1.9. Let W be an incompressible subsurface of S. Then consider Q_W, the space of integrable holomorphic quadratic differentials on W, where we have chosen a complex structure on the smooth surface W such that neighborhoods of boundary components of ∂W are conformally punctured disks. To any such quadratic differential q, the corresponding singular flat metric on W thus assigns length zero to any peripheral curve. Let λ be a measured lamination supported on the complement $S \setminus W$. The triple (W, q, λ) is called a *mixed structure* on S.

To any $\eta = (W, q, \lambda)$, one obtains a geodesic current L_η given by the property

$$i(L_\eta, \gamma) = i(\lambda, \gamma) + \frac{1}{2}\int_0^{\pi/2} i(v_q^{\theta}, \gamma)\, d\theta,$$

where λ is a closed curve on S. We remark that in the case $W = \emptyset$, then η is a measured lamination on S, so that the space $\mathrm{Mix}(S)$ properly contains $\mathrm{ML}(S)$.

The compactification of the singular flat metrics arising from unit-norm quadratic differentials is then given by the following theorem.

Theorem 1.10 (Duchin-Leininger-Rafi [16])**.** *The closure of* $\mathrm{Flat}(S)$ *in* $\mathrm{PCurr}(S)$ *is given by* $\mathrm{Flat}(S) \sqcup \mathrm{PMix}(S)$.

1.3.5 Harmonic maps between surfaces

Let $(M, \sigma|dz|^2)$ and $(N, \rho|dw|^2)$ be two closed Riemannian surfaces and $w : (M, \sigma|dz|^2) \to (N, \rho|dw|^2)$ a Lipschitz map. Then the energy of the map w is given by the integral

$$\mathcal{E}(w) := \frac{1}{2}\int_M ||dw||^2 \, dvol_\sigma.$$

A critical point of the energy functional is a *harmonic map.* We remark that if the domain M is a surface, the energy is a conformal invariant; hence a harmonic map depends only upon the conformal class of the domain but depends on the metric of the target surface. The energy density of the map w at a point is given by

$$e(w) = \frac{\rho(w(z))}{\sigma(z)}(|w_z|^2 + |w_{\overline{z}}|^2),$$

and so the total energy is also given by the formula

$$\begin{aligned} \mathcal{E}(w) &= \int_M e(w)\, \sigma \, dz \wedge d\overline{z} \\ &= \int_M \rho(w(z))(|w_z|^2 + |w_{\overline{z}}|^2)\, dz \wedge d\overline{z}, \end{aligned}$$

once again seeing that the total energy depends only upon the conformal structure of the domain and the metric of the target. Alternatively, a harmonic map w solves the Euler-Lagrange equation for the energy functional, a second-order nonlinear PDE:

$$w_{z\overline{z}} + (\log \rho)_w w_z w_{\overline{z}} = 0.$$

To any harmonic map $w : (M, \sigma|dz|^2) \to (N, \rho|dw|^2)$, the pull-back of the metric tensor decomposes by type according to

$$w^*\rho = \Phi dz^2 + \sigma e dz d\overline{z} + \overline{\Phi} d\overline{z}^2,$$

where $\Phi\, dz^2$ is a holomorphic quadratic differential with respect to the complex structure coming from the conformal class of $(M, \sigma|dz|^2)$ called the *Hopf differential* of w. Much of the formulas arising from harmonic maps make use of the auxiliary functions:

$$\mathcal{H} = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2$$
$$\mathcal{L} = \frac{\rho(w(z))}{\sigma(z)}|w_{\overline{z}}|^2.$$

We list some of these formulas and make liberal use of them without always explicitly citing the precise one:

The energy density $e = \mathcal{H} + \mathcal{L}$

The Jacobian $\mathcal{J} = \mathcal{H} - \mathcal{L}$

The norm of the quadratic differential $|\Phi|^2/\sigma^2 = \mathcal{H}\mathcal{L}$

The Lapace-Beltrami operator $\Delta \equiv \dfrac{4}{\sigma}\dfrac{\partial^2}{\partial z \partial \overline{z}}$

Gaussian curvature of the source $K(\sigma) = -\dfrac{2}{\sigma}\dfrac{\partial^2 \log \sigma}{\partial z \partial \overline{z}}$

Gaussian curvature of the target $K(\rho) = -\dfrac{2}{\rho}\dfrac{\partial^2 \log \rho}{\partial w \partial \overline{w}}$

The Beltrami differential $\nu = \dfrac{w_{\overline{z}}}{w_z} = \dfrac{\overline{\Phi}}{\sigma\mathcal{H}}$ and $|\nu|^2 = \dfrac{\mathcal{L}}{\mathcal{H}}$.

The Bochner formula is given by

$$\Delta \log \mathcal{H} = -2K(\rho)\mathcal{H} + 2K(\rho)\mathcal{L} + 2K(\sigma), \qquad \text{when } \mathcal{H}(p) \neq 0$$

$$\Delta \log \mathcal{L} = -2K(\rho)\mathcal{L} + 2K(\rho)\mathcal{H} + 2K(\sigma), \qquad \text{when } \mathcal{L}(p) \neq 0.$$

We shall often be in the setting where both the source and target are hyperbolic surfaces, that is $K(\sigma) = K(\rho) \equiv -1$, and so some of the formulas listed above can be simplified. In the more general setting where the target has negative curvature, the existence of a harmonic map in the homotopy class is due to Eells-Sampson [18], its uniqueness is due to Hartman [29] and Al'ber [1], and that if the homotopy class contains a diffeomorphism, then the harmonic map itself is a diffeomorphism and $\mathcal{H} > 0$ is due to Schoen-Yau [56] and Sampson [52].

1.4 Higher Teichmüller theory

It has been discussed in the previous section that Teichmüller space may be regarded as a space of conjugacy classes of discrete and faithful representations into $\mathrm{PSL}(2,\mathbb{R})$. From an algebraic and topological perspective, one may be interested in the representation variety

$$\chi(G) := \mathrm{Hom}(\pi_1(S), G)//G,$$

the space of conjugacy classes of surface group representations into a Lie group. Seminal work of Goldman [26], reveals that in the case where $G = \mathrm{PSL}(2,\mathbb{R})$, the representation variety has precisely $4g - 3$ connected components, each indexed by a topological invariant known as the *Euler class*. The components which attain the maximal Euler class $2g - 2$ and $2 - 2g$ are two copies of Teichmüller space, $\mathrm{Teich}(S)$ and $\mathrm{Teich}(\overline{S})$, respectively, where $\overline{S}$ is the surface S with the opposite orientation.

The goal to ascertain the topology of distinguished components of representation varieties into higher rank Lie groups is the genesis of the higher rank Teichmüller theory. In foundational work by Hitchin [31], the number of the connected components of the $\mathrm{PSL}(n,\mathbb{R})$ representation variety is given explicitly, and is shown to depend solely on the parity of n.

Theorem 1.11 (Hitchin [31]). *The space* $\mathrm{Hom}^+(\pi_1(S),\mathrm{PSL}(n,\mathbb{R}))/\mathrm{PSL}(n,\mathbb{R})$ *has, for* $n>2$*, three connected components if* n *is odd, and six components if* n *is even.*

The Teichmüller portion of higher Teichmüller theory is not entirely arbitrary. Classical representation theory shows that given $\mathrm{PSL}(n,\mathbb{R}), n>2$, there is a unique irreducible representation $\iota : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$, so that in these representation varieties, there is a distinguished component which contains the image of Teichmüller space under the map

$$I : \mathrm{Teich}(S) \hookrightarrow \mathrm{Hom}^+(\pi_1(S),\mathrm{PSL}(n,\mathbb{R}))/\mathrm{PSL}(n,\mathbb{R})$$

$$\rho \mapsto \iota \circ \rho.$$

The connected component which contains this copy of Teichmüller space is called the *Hitchin component*, and its topology is completely understood.

Theorem 1.12 (Hitchin [31]). *The Hitchin component* $\mathrm{Hit}_n(S)$ *is homeomorphic to* $\mathbb{R}^{(2g-2)(n^2-1)}$.

Hitchin proves these two results using *Higgs bundles*, which will be discussed in the following section. However, the analytic approach via Higgs bundles reveal no geometric significance of the underlying objects. With Teichmüller space, though it may be seen as a representation variety, the underlying objects may also be viewed as hyperbolic metrics. In the case when $n = 3$, work by Goldman [27] and Choi-Goldman [8] shows the Hitchin component for $\mathrm{PSL}(3, \mathbb{R})$ parameterizes the space of *convex real projective structures* on S. These are projective structures on S, whose developed image is a strictly convex subset of $\mathbb{RP}^2$.

1.4.1 Nonabelian Hodge correspondence

The nonabelian Hodge correspondence furnishes a dictionary between three particular moduli spaces. The first is known as the *Betti moduli space* and is the representation variety discussed previously

$$\mathrm{Hom}(\pi_1(S), G)//G,$$

for a Lie group G, where the double slash denotes the GIT quotient (an algebraic quotient to ensure the resulting space is Hausdorff). However, the usual quotient by taking conjugacy classes is sufficient when $G = \mathrm{PSL}(2, \mathbb{R})$, and the representation is discrete and faithful. For the remainder of the section however, for simplicity, we describe the correspondence for $G = \mathrm{SL}(n, \mathbb{C})$. For other groups, additional conditions need to be imposed on the objects.

The Betti moduli space has the structure of an algebraic variety, as the space is determined explicitly by the data of where the usual $2g$ generators of the fundamental group are sent to in G, with the one algebraic relation given by the product of the commutators being the identity element of G.

The second moduli space, the *deRham moduli space* is the space of flat connections on rank n complex vector bundles. These first two moduli spaces are diffeomorphic and the correspondence has the historical name of the *Riemann-Hilbert correspondence*. To obtain a flat connection from a representation ρ, one constructs the flat bundle $\widetilde{S} \times_\rho \mathbb{C}^n$, which will have a natural flat structure, and hence a flat connection. From a flat bundle, one takes the monodromy representation to recover the representation.

The third moduli space, called the *Dolbeault moduli space*, features holomorphic objects known as *Higgs bundles* on a Riemann surface, which are given by the data of a holomorphic vector bundle $\mathcal{E}$ and a *Higgs field* ϕ, which is a holomorphic $(1,0)$-form with values in $\operatorname{End}(\mathcal{E})$. An additional property of the Higgs bundle is required, and that is the notion of stability. A *stable* Higgs bundle is a Higgs bundle, where for any ϕ-invariant nonzero proper subbundle $\mathcal{F} \subset \mathcal{E}$, one has that the slope condition

$$\frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}} < \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}$$

is satisfied.

The correspondence between the Dolbeault moduli space with the Betti and deRham moduli spaces is due to the combined independent work of Donaldson [15], Corlette [10], Hitchin [30] and Simpson [54]. The dictionary is completed by showing existence of a ρ-equivariant harmonic maps from the universal cover of S to the associated symmetric space of G.

1.4.2 Labourie conjecture

As noted before, a Higgs bundle requires the choice of a Riemann surface to make sense of holomorphicity. Yet, in the original data of a representation, there is only topological data coming from the smooth structure of the surface (recall we only make use of the Lie group and the fundamental group $\pi_1(S)$ of the surface S to define a representation). Hence, in the nonabelian Hodge correspondence, a choice of a Riemann surface structure must be made. The correspondence holds for any choice of complex structure endowed on S. The issue here is that the representation variety is acted upon by the mapping class group (the action is on $\pi_1(S)$), whereas for a Higgs bundle, the action is not evident. To this end, Labourie has conjectured there is an apt choice of a complex structure for each such representation.

Conjecture 1.13. *If ρ is a Hitchin representation into a real split Lie group G, then there is a unique complex structure X on S such that the ρ-equivariant harmonic map from X to G/K is weakly conformal.*

The conjecture has been fully resolved in the case of real rank 2 split Lie groups.

Theorem 1.14 (Labourie [38])**.** *Let ρ be a Hitchin representation for* $G = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{Sp}(4,\mathbb{R})$, *or G_2, then there is a unique complex structure X on S such that the ρ-equivariant harmonic map from X to G/K is weakly conformal.*

As a historical note, the case when $G = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ was first proven by Schoen [53], in the context of studying minimal lagrangians on the bidisc as a way to give a midpoint map on Teichmüller space.

The Labourie theorem gives a parametrization of the groups G listed above by a bundle of differentials over Teichmüller space. In the order written above, the Hitchin component of the G representation variety are parameterized by the bundle of quadratic, cubic (shown independently by Loftin [40]), quartic and sextic differentials over Teichmüller space.

Chapter 2

Embedding of induced metrics

In this chapter, we discuss the metric structure of the minimal lagrangians in the bidisc. In particular, we discuss the relation between the Labourie parameterization of Hitchin $\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$ representations by the bundle of quadratic differentials over Teichmüller space with geometric properties of the minimal lagrangians in $\mathbb{H}^2 \times \mathbb{H}^2$. In the process, we present two new results in the classical theory of harmonic maps between surfaces.

2.1 Minimal lagrangians

A diffeomorphism $\phi : (S, g_1) \to (S, g_2)$ is said to be minimal if its graph $\Sigma \subset (S \times S, g_1 \oplus g_2)$ with the induced metric is a minimal surface. Observe that if ϕ is minimal then so is ϕ^{-1}. If ω_1 and ω_2 denote the area forms of g_1, g_2 respectively, and if in addition $\Sigma \subset (S\times S, \omega_1 - \omega_2)$ is a Lagrangian submanifold, then we say ϕ is a minimal lagrangian.

Theorem 2.1 (Schoen [53])**.** *If g_1 and g_2 are hyperbolic metrics on S, then there is a unique minimal lagrangian map $\phi : (S, g_1) \to (S, g_2)$ in the homotopy class of the identity.*

Let Σ denote the graph of ϕ, which is a minimal surface, with the induced metric. Then its inclusion into the product $i : \Sigma \to (S \times S, g_1 \oplus g_2)$ is a conformal harmonic map. A conformal harmonic map to a product space is an ordered pair of harmonic maps whose Hopf differentials sum to zero. Hence to any pair of points in Teichmüller space, one may record the data of both the conformal structure of the minimal surface along with one of the Hopf differentials. The harmonic maps parametrization of Teichmüller space which we record below ensures the map is bijective. Sampson proved injectivity and continuity of the map, and Wolf showed the map was surjective and admits a continuous inverse.

Theorem 2.2 (Sampson [52], Wolf [61])**.** *Let (S, σ) be a fixed hyperbolic surface. To any point in Teichmüller space $[(S, \rho)]$, select the representative (S, ρ) so that the identity map $id : (S, \sigma) \to (S, \rho)$ is the unique harmonic map in its homotopy class and denote its Hopf differential $\Phi(\rho)$. Then this map*

$$\Phi : \mathcal{T}(S) \to H^0(X, K_X^2)$$

is a homeomorphism, where X is the complex structure associated to (S, σ).

Theorem 2.3. *The map*

$$\Psi : \mathcal{T}(S) \times \mathcal{T}(S) \to \ Q_g$$

$$(X_1, X_2) \mapsto \mathrm{Hopf}(u_1)$$

which assigns to any pair of points X_1, X_2 in Teichmüller space, the conformal structure of the unique graph minimal surface $\Sigma \subset X_1 \times X_2$ along with the Hopf differential Hopf(u_1) of the projection $u_1 : \Sigma \to X_1$ is a homeomorphism.

Proof. The discussion above ensures the map Ψ is well-defined. As the construction of the minimal surface varies continuously with the choice of X_1, X_2, it is clear the map is continuous. To see injectivity of Ψ, suppose that $\Psi(X_1, X_2) = \Psi(Y_1, Y_2) = (\Sigma, \Phi)$. Then the harmonic maps $u_1 : \Sigma \to X_1$ and $v_1 : \Sigma \to Y_1$ have the same Hopf differentials, so by the harmonic maps parameterization, $X_1 = Y_1$. The same argument forces $X_2 = Y_2$. Surjectivity follows similarly, as to any choice of Riemann surface $\Sigma = (S, J)$ and holomorphic quadratic differential Φ, there exists a unique hyperbolic metric $X_1 = (S, g_1)$, so that the identity map $id : \Sigma \to X_1$ is a harmonic map with Hopf differential Φ. Similarly one can find an X_2 arising from the Hopf differential $-\Phi$. Hence $\Psi(X_1, X_2) = (\Sigma, \Phi)$ which gives surjectivity. The inverse is clearly continuous as given the data of a Riemann surface and a holomorphic quadratic differential, the pair of hyperbolic metrics may be written explicitly and vary continuously, which suffices for the proof. □

2.1.1 Embedding

In this section we study the induced metric on the graph minimal surfaces. Recall that given an ordered pair (X_1, X_2) of hyperbolic surfaces, Theorem 2.1 produces a graph minimal surface Σ in the 4-manifold $(S \times S, g_1 \oplus g_2)$, where $X_i = (S, g_i)$. If $m : (S, g_1) \to (S, g_2)$ is the unique minimal map isotopic to the identity, then $id : (S, g_1) \to (S, m^* g_2)$ is the unique minimal map isotopic to the identity, which in this case is the identity map. The graph Σ then, is the diagonal in $S \times S$ and there is a canonical diffeomorphism from the surface S to the diagonal in $S \times S$. The induced metric on Σ thus furnishes a metric g on S by the pullback of this diffeomorphism. Henceforth, when we say *induced metric*, we refer to this metric g on S, and will use Σ to denote (S, g). We consider these metrics up to pullback by a diffeomorphism isotopic to the identity, and call this subspace of metrics $\mathrm{Ind}(S)$ and endowing it with the compact-open topology. The remainder of the section is devoted towards studying geometric properties of the minimal surfaces and showing $\mathrm{Ind}(S)$ can be embedded into $\mathrm{PCurr}(S)$.

Proposition 2.4. *Let* $X_1 = (S, g_1), X_2 = (S, g_2)$ *and* $\Psi(X_1, X_2) = (\Sigma, \Phi)$. *Then the induced metric on the minimal surface* Σ *is given by* $g_1 + m^* g_2$. *Consequently, the induced metric is given by twice the* $(1,1)$ *part of a hyperbolic metric when expressed in conformal coordinates.*

Proof. As in the discussion above, we may choose a suitable hyperbolic metric $X_2 = (S, g_2)$ in the equivalence class of $[X_2]$ to ensure the unique minimal map isotopic to the identity is the identity map. Hence, the graph of the minimal map is the diagonal in $S \times S$, so that (after identifying the diagonal with S) the harmonic map from the minimal surface Σ to X_i is given by the identity map. The first result then follows by definition of the product metric. Notice that the hyperbolic metric g_1 may be written in conformal coordinates on Σ as $\Phi dz^2 + \sigma e dz d\bar{z} + \overline{\Phi} d\bar{z}^2$. As the minimal surface Σ is mapped conformally into the product $X_1 \times X_2$ of hyperbolic surfaces, then one obtains a pair $u_i : \Sigma \to X_i$ of harmonic maps, whose Hopf differentials, Hopf(u_1) and Hopf(u_2), sum to zero. Hence g_2 may be written in conformal coordinates on Σ as $-\Phi dz^2 + \sigma e dz d\bar{z} - \overline{\Phi} d\bar{z}^2$, for $|\Phi| = |-\Phi|$, so by a result of Sampson (Proposition 2.7), the energy densities will coincide. As the induced metric is given by the sum, the induced metric has local expression $2\sigma e dz d\bar{z}$. □

Proposition 2.5. *The induced minimal surfaces have strictly negative sectional curvature.*

Proof. For any point $p \in \Sigma$, it is clear that $K_p \leq 0$, as Σ is a minimal surface in a NPC space, so we wish to show that $K_p \neq 0$. The proof is by contradiction. Let $\{e_1, e_2\}$ be an orthonormal basis of $N_p\Sigma$. Now consider the 2-plane spanned by eigenvectors X and Y of the second fundamental form II. Then one has $\mathrm{II}(X, Y) = \sum_{j=1}^{2} \mathrm{II}_j(X, Y) e_j$. Then the mean curvatures of the immersion are given by

$$H_1 = \mathrm{II}_1(X, X) + \mathrm{II}_1(Y, Y) = 0 \tag{2.1}$$

$$H_2 = \mathrm{II}_2(X, X) + \mathrm{II}_2(Y, Y) = 0 \tag{2.2}$$

Then the Gauss equation tells us that at p,

$$0 = Rm(X,Y,Y,X) = \widetilde{Rm}(X,Y,Y,X) - \langle \mathrm{II}(X,X), \mathrm{II}(Y,Y)\rangle + \langle \mathrm{II}(X,Y), \mathrm{II}(X,Y)\rangle \tag{2.3}$$

$$= \widetilde{Rm}(X,Y,Y,X) + \sum_{j=1}^{2} \mathrm{II}_j(X,X)\mathrm{II}_j(Y,Y) - \sum_{j=1}^{2} \mathrm{II}_j(X,Y)^2, \tag{2.4}$$

and as $\mathbb{H}^2 \times \mathbb{H}^2$ is NPC, from (4.1), (4.2) and (4.4), it follows $\mathrm{II} \equiv 0$ at p and that $\widetilde{Rm}(X,Y,Y,X) = 0$ at p. As $T(\mathbb{H}^2 \times \mathbb{H}^2) \cong T\mathbb{H}^2 \oplus T\mathbb{H}^2$, we may write $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. A simple calculation shows:

$$\begin{aligned} 0 = \widetilde{Rm}(X,Y,Y,X) &= Rm_1(X_1,Y_1,Y_1,X_1) + Rm_2(X_2,Y_2,Y_2,X_2) \\ &= \kappa(X_1,Y_1)\left(|X_1|^2|Y_1|^2 - \langle X_1,Y_1\rangle^2\right) + \kappa(X_2,Y_2)\left(|X_2|^2|Y_2|^2 - \langle X_2,Y_2\rangle^2\right) \\ &= -1\cdot\left(|X_1|^2|Y_1|^2 - \langle X_1,Y_1\rangle^2\right) - 1\cdot\left(|X_2|^2|Y_2|^2 - \langle X_2,Y_2\rangle^2\right), \end{aligned}$$

which by Cauchy-Schwarz implies that X_1 and Y_1 (and also X_2 and Y_2) are linearly dependent, so that the map u_{1_*} drops rank, a contradiction, as our surface was a graph. □

Proposition 2.6. *The second fundamental form is given by*

$$\begin{aligned}
\mathrm{II}(E_1, E_1) &= \frac{-\operatorname{Re}\Phi(\sigma e)_y - \sigma e(\operatorname{Im}\Phi)_x + \operatorname{Im}\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_1 \\
&\quad + \frac{\operatorname{Im}\Phi(\sigma e)_y - \sigma e(\operatorname{Re}\Phi)_x + \operatorname{Re}\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_2 \\
\mathrm{II}(E_2, E_2) &= \frac{\operatorname{Re}\Phi(\sigma e)_y + \sigma e(\operatorname{Im}\Phi)_x - \operatorname{Im}\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_1 \\
&\quad + \frac{-\operatorname{Im}\Phi(\sigma e)_y + \sigma e(\operatorname{Re}\Phi)_x - \operatorname{Re}\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_2 \\
\mathrm{II}(E_1, E_2) &= \frac{\operatorname{Im}\Phi(\sigma e)_y - \sigma e(\operatorname{Re}\Phi)_x + \operatorname{Re}\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_1 \\
&\quad + \frac{-\sigma e(\operatorname{Re}\Phi)_y + \operatorname{Re}\Phi(\sigma e)_y - \operatorname{Im}\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_2.
\end{aligned}$$

Proof. For a choice of complex coordinates $z = x + iy$ on the minimal surface Σ, then $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ is an orthogonal frame. Denote then $E_1 = \frac{\partial}{\partial x}/|\frac{\partial}{\partial x}|_\Sigma$ and $E_2 = \frac{\partial}{\partial y}/|\frac{\partial}{\partial y}|_\Sigma$. Let J be the almost complex structure on the 4-manifold $X_1 \times X_2$, then $J = J_1 \oplus J_2$, where J_i is the almost complex structure arising from $X_i = (S, g_i)$. As $\Sigma \subset X_1 \times X_2$ is a lagrangian submanifold, then $\{E_1, E_2, JE_1, JE_2\}$ forms an orthonormal basis of $T(X_1 \times X_2) \cong TX_1 \oplus TX_2$ in this neighborhood. The second fundamental form then is given by

$$\mathrm{II}(X, Y) = \sum_{j=1}^{2} \widetilde{g}(\widetilde{\nabla}_X Y, JE_j) JE_j,$$

where $\widetilde{g} = g_1 \oplus g_2$ and $\widetilde{\nabla} = \nabla_1 \oplus \nabla_2$. We first calculate $\mathrm{II}(E_1, E_1)$. As the minimal surface metric is given by $2\sigma e|dz|^2 = 2\sigma e(dx^2 + dy^2)$, one has

$$2\sigma e(dx^2 + dy^2)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = 2\sigma e = \left\|\frac{\partial}{\partial x}\right\|_\Sigma^2,$$

so that

$$E_1 = \frac{\frac{\partial}{\partial x}}{\sqrt{2\sigma e}}.$$

Similarly E_2 is given by

$$E_2 = \frac{\frac{\partial}{\partial y}}{\sqrt{2\sigma e}}.$$

To calculate JE_1, we project E_1 to each of its factors and apply the almost complex structure on each of its factors, namely, we find the vector which has the same length and forms angle $\pi/2$ with the projected factor using the hyperbolic metric. This is the complex structure arising from the conformal class of the metric. To find $J_1E_1 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ for instance, we observe first the hyperbolic metric on X_1 is given by

$$\rho_1 = \Phi dz^2 + \sigma e dz d\overline{z} + \overline{\Phi} d\overline{z}^2 = (2\operatorname{Re}\Phi + \sigma e)dx^2 - 4\operatorname{Im}\Phi dxdy + (-2\operatorname{Re}\Phi + \sigma e)dy^2$$

Hence we want to solve $a \neq 0$, $b > 0$ for which

$$g_1\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, E_1\right) = 0 \tag{2.5}$$

$$g_1\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = g_1(E_1, E_1) = \frac{2\operatorname{Re}\Phi + \sigma e}{2\sigma e}. \tag{2.6}$$

Some basic algebra yields that $a = \frac{2\operatorname{Im}\Phi}{\sqrt{(2\sigma e)((\sigma e)^2 - 4|\Phi|^2)}}$ and $b = \frac{2\operatorname{Re}\Phi + \sigma e}{\sqrt{(2\sigma e)((\sigma e)^2 - 4|\Phi|^2)}}$, so that

$$J_1E_1 = \frac{2\operatorname{Im}\Phi}{\sqrt{(2\sigma e)((\sigma e)^2 - 4|\Phi|^2)}}\frac{\partial}{\partial x} + \frac{2\operatorname{Re}\Phi + \sigma e}{\sqrt{(2\sigma e)((\sigma e)^2 - 4|\Phi|^2)}}\frac{\partial}{\partial y}.$$

Notice that the denominator appearing in b is positive, as $2|\Phi| < \sigma e$ when $\Phi \equiv 0$, in which case the minimal surface is a totally geodesic subsurface. Now J_2E_1 is found similarly, and is given by

$$J_2E_1 = \frac{-2\operatorname{Im}\Phi}{\sqrt{2\sigma e((\sigma e)^2 - 4|\Phi|^2)}}\frac{\partial}{\partial x} + \frac{-2\operatorname{Re}\Phi + \sigma e}{\sqrt{2\sigma e((\sigma e)^2 - 4|\Phi|^2)}}\frac{\partial}{\partial y}.$$

The tangent vector given by $\widetilde{\nabla}_{E_1}E_1$ splits as $\nabla^1_{E_1}E_1 \oplus \nabla^2_{E_1}E_1$. The Christoffel symbols for g_1 and g_2 can be readily calculated.

$$\begin{aligned}
\nabla^1_{E1}E_1 &= \nabla^1_{\frac{\frac{\partial}{\partial x}}{\sqrt{2\sigma e}}} \frac{\frac{\partial}{\partial x}}{\sqrt{2\sigma e}} \\
&= \frac{1}{\sqrt{2\sigma e}}\left(\frac{1}{\sqrt{2\sigma e}}\nabla^1_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} + \left(\frac{1}{\sqrt{2\sigma e}}\right)_x \frac{\partial}{\partial x}\right) \\
&= \frac{1}{\sqrt{2\sigma e}}\left(\frac{1}{\sqrt{2\sigma e}}\left({}^1\Gamma^1_{11}\frac{\partial}{\partial x} +^1 \Gamma^2_{11}\frac{\partial}{\partial y}\right) + \left(\frac{1}{\sqrt{2\sigma e}}\right)_x \frac{\partial}{\partial x}\right), \\
&= \left(\frac{1}{2\sigma e}{}^1\Gamma^1_{11} + \frac{1}{\sqrt{2\sigma e}}\left(\frac{1}{\sqrt{2\sigma e}}\right)_x\right)\frac{\partial}{\partial x} + \frac{1}{2\sigma e}{}^2\Gamma^2_{11}\frac{\partial}{\partial y}
\end{aligned}$$

where ${}^1\Gamma^1_{11}$ and ${}^1\Gamma^2_{11}$ are the usual Christoffel symbols, where the extra superscript denotes these are the ones for the metric g_1. There are given explicitly by

$$\begin{aligned}
{}^1\Gamma^1_{11} &= \frac{1}{2}\left(\frac{-2\operatorname{Re}\Phi + \sigma e}{\sigma^2e^2 - 4|\Phi|^2}(2\operatorname{Re}\Phi + \sigma e)_x + \frac{2\operatorname{Im}\Phi}{\sigma^2e^2 - 4|\Phi|^2}\left((-4\operatorname{Im}\Phi_x) - (2\operatorname{Re}\Phi + \sigma e)_y\right)\right) \\
{}^2\Gamma^1_{11} &= \frac{1}{2}\left(\frac{2\operatorname{Im}\Phi}{\sigma^2e^2 - 4|\Phi|^2}(2\operatorname{Re}\Phi + \sigma e)_x + \frac{2\operatorname{Re}\Phi + \sigma e}{\sigma^2e^2 - 4|\Phi|^2}\left((-4\operatorname{Im}\Phi_x) - (2\operatorname{Re}\Phi + \sigma e)_y\right)\right).
\end{aligned}$$

Similarly, the same can be done for the metric g_2 and using the formula for II, one gets $\mathrm{II}(E_1, E_1)$. The same can be done for the rest. □

It would be curious to see under what conditions different points in Q_g would yield the same induced metric. One might hope that the space of induced metrics would be homeomorphic to Q_g, but the following result of Sampson shows this is not possible:

Proposition 2.7 (Sampson)**.** *For a fixed closed hyperbolic surface $X = (S, \sigma)$, if Φ_1 and Φ_2 are two Hopf differentials on X arising from harmonic maps from X to closed* *hyperbolic surfaces of the same genus, such that the norms* $|\Phi_1|$ *and* $|\Phi_2|$ *coincide, then the energy densities coincide, that is* $e_1 = e_2$.

Hence, if we select two elements of Q_g, say (X, Φ_1) and (X, Φ_2), where $|\Phi_1| = |\Phi_2|$, but $\Phi_1 \neq \Phi_2$, then the corresponding energy densities are the same and hence the corresponding induced metrics are the same.

The following proposition is a converse to the result of Sampson and shows this is the only situation for which the corresponding induced metrics coincide.

Lemma 2.8. *On a fixed closed hyperbolic surface, we have* $e_1 = e_2$ *if and only if* $|\Phi_1| = |\Phi_2|$.

Proof. That $|\Phi_1| = |\Phi_2|$ implies $e_1 = e_2$ is due to Sampson. Now suppose $e_1 = e_2$, then $\mathcal{H}_1 + \mathcal{L}_1 = \mathcal{H}_2 + \mathcal{L}_2$, so the Bochner formula $\triangle \log \mathcal{H}_i = 2\mathcal{H}_i - 2\mathcal{L}_i - 2$ may be rewritten as $\triangle \log \mathcal{H}_i = 4\mathcal{H}_i - 2e_i - 2$. Subtracting the two equations for $i = 1, 2$ yields

$$\triangle \log \frac{\mathcal{H}_1}{\mathcal{H}_2} = 4(\mathcal{H}_1 - \mathcal{H}_2).$$

Now $\mathcal{H}_i > 0$, so that the quotient $\mathcal{H}_1/\mathcal{H}_2$ attains its maximum on the surface, which we claim is 1, for if the maximum of $\mathcal{H}_1/\mathcal{H}_2$ is greater than 1, then at the maximum (which is also the maximum of $\log \frac{\mathcal{H}_1}{\mathcal{H}_2}$)

$$0 \geq \triangle \log \frac{\mathcal{H}_1}{\mathcal{H}_2} = 4(\mathcal{H}_1 - \mathcal{H}_2) = 4\mathcal{H}_2 \left(\frac{\mathcal{H}_1}{\mathcal{H}_2} - 1 \right) > 0,$$

a contradiction, so that $\frac{\mathcal{H}_1}{\mathcal{H}_2} \leq 1$ and symmetrically $\frac{\mathcal{H}_2}{\mathcal{H}_1} \leq 1$, hence $\mathcal{H}_1 = \mathcal{H}_2$ and so $\mathcal{L}_1 = \mathcal{L}_2$, by the assumption on the energy densities. From the formula $|\Phi|^2/\sigma^2 = \mathcal{H}\mathcal{L}$, the conclusion follows.

□

Corollary 2.9. *The space of induced metrics* $\mathrm{Ind}(S)$ *may be identified with* $Q_g/\sim$, *where* $(X, \Phi_1) \sim (Y, \Phi_2)$ *if* $X = Y$ *and* $|\Phi_1| = |\Phi_2|$.

We conclude this section by proving the space $\mathrm{Ind}(S)$ can be embedded into the space of currents and that the embedding remains injective after projectivization, thereby obtaining an embedding into projectivized currents.

Proposition 2.10. *The space* $\mathrm{Ind}(S)$ *can be realized as geodesic currents.*

Proof. From Proposition 2.5, the induced metrics have strictly negative curvature, so by Otal [47], there is a well-defined embedding $\mathcal{C} : \mathrm{Ind}(S) \to \mathrm{Curr}(S)$, from the space of induced metrics on S to the space of geodesic currents, which sends $2\sigma e \mapsto L_{2\sigma e}$, so that if γ is a closed curve, then $l_{2\sigma e}([\gamma]) = i(L_{2\sigma e}, \gamma)$. □

The following lemma is a statement concerning energy densities and their failure to scale linearly.

Lemma 2.11. *On a fixed closed hyperbolic surface, if* $e_1 = ce_2$, *then* $c = 1$, *and hence* $|\Phi_1| = |\Phi_2|$.

Proof. Without loss of generality, suppose $c \geq 1$, else we may reindex so that $c \geq 1$. Then $\frac{\mathcal{H}_1}{\mathcal{H}_2} \leq c$, for if $\frac{\mathcal{H}_1}{\mathcal{H}_2} > c$, we locate the maximum of $\mathcal{H}_1/\mathcal{H}_2$, and the Bochner formula at that point yields

$$
\begin{aligned}
0 \geq \triangle \log \frac{\mathcal{H}_1}{\mathcal{H}_2} &= 4(\mathcal{H}_1 - \mathcal{H}_2) - 2(e_1 - e_2) \\
&= 4(\mathcal{H}_1 - \mathcal{H}_2) - 2(ce_2 - e_2) \\
&= 4\mathcal{H}_2 \left(\frac{\mathcal{H}_1}{\mathcal{H}_2} - 1 \right) - 2e_2 \, (c-1) \\
&> 4\mathcal{H}_2 \, (c-1) - 2e_2 \, (c-1) \\
&= (c-1)(4\mathcal{H}_2 - 2e_2) \\
&= (c-1)(2\mathcal{H}_2 - 2\mathcal{L}_2) = 2(c-1) \, \mathcal{J}_2 > 0,
\end{aligned}
$$

a contradiction. Notice the upper bound is actually attained, for at a zero of $|\Phi_1|$, we have that $\mathcal{L}_1$ vanishes and so at such a zero we have the equation

$$\mathcal{H}_1 = c\mathcal{H}_2 + c\mathcal{L}_2,$$

and as we have $\mathcal{H}_1/\mathcal{H}_2 \leq c$, it follows that $\mathcal{L}_2$ must also vanish whenever $\mathcal{L}_1$ does. In fact, we can say more about the zeros of $\mathcal{L}_i$. The condition on the energy densities yields the equality

$$0 = (c\mathcal{H}_2 - \mathcal{H}_1) + (c\mathcal{L}_2 - \mathcal{L}_1),$$

and the bound on the quotient $\mathcal{H}_1/\mathcal{H}_2$ implies that the first term is nonnegative so the second term is nonpositive, that is $c\mathcal{L}_2 - \mathcal{L}_1 \leq 0$ or $c \leq \mathcal{L}_1/\mathcal{L}_2$ or $\mathcal{L}_2/\mathcal{L}_1 \leq 1/c$, so that the order of the zeros of $\mathcal{L}_2$ is greater than or equal to the order of zeros of $\mathcal{L}_1$. As $|\Phi|^2/\sigma^2 = \mathcal{H}\mathcal{L}$ and $\mathcal{H} > 0$, then both $\mathcal{L}_1$ and $\mathcal{L}_2$ have exactly $8g - 8$ zeros counted with multiplicity, so that the order of vanishing of $\mathcal{L}_1$ is the same as that of $\mathcal{L}_2$ at every point of the surface. Hence the quadratic differentials Φ_1 and Φ_2 differ by a multiplicative constant $k \in \mathbb{C}$, that is $\Phi_1 = k\Phi_2$. At the zero of $|\Phi_2|$ (and so also a zero of $|\Phi_1|$), which is a maximum of the quotient $\mathcal{H}_1/\mathcal{H}_2$, the Bochner equation now reads,

$$\begin{aligned} 0 \geq \triangle \log \frac{\mathcal{H}_1}{\mathcal{H}_2} &= 2\mathcal{H}_1 - \frac{2|\Phi_1|^2}{\sigma^2\mathcal{H}_1} - 2\mathcal{H}_2 + \frac{2|\Phi_2|^2}{\sigma^2\mathcal{H}_2} \\ &= 2(\mathcal{H}_1 - \mathcal{H}_2) \\ &= 2\mathcal{H}_2(c-1) \geq 0, \end{aligned}$$

which implies $c = 1$, and by the previous lemma $|k| = 1$.

□

Theorem 2.12. *The space of induced metrics* $\mathrm{Ind}(S)$ *embeds into* $\mathrm{PCurr}(S)$*.*

Proof. Let $\pi : \mathrm{Curr}(S) \to \mathrm{PCurr}(S)$ be the natural projection map. It suffices to show the map $\pi \circ \mathcal{C} : \mathrm{Ind}(S) \to \mathrm{PCurr}(S)$ is injective. If the image of two induced metrics under the map $\pi \circ \mathcal{C}$ coincide, that is $\sigma dz d\bar{z} = c\sigma' e' dz d\bar{z}$, where $c \in \mathbb{R}_{>0}$, then they will be in the same conformal class, so that $\sigma = \sigma'$. Then $e = ce'$, and by Lemma 2.11, $c = 1$.

□

Remark 2.13. As the induced metrics are not scalar multiples of each other, we make a slight modification by dividing the induced metrics by 2 to ensure these metrics are now precisely the $(1,1)$-part of a hyperbolic metric when written in conformal coordinates rather than twice that.

Chapter 3

Compactification

In this chapter, we identify the elements in the closure $\overline{\mathrm{Ind}(S)} \subset \mathrm{PCurr}(S)$. As the space of projectivized currents is compact, we obtain a compactification $\mathrm{Ind}(S) \sqcup \mathrm{PMix}(S)$ of the induced metrics from the embedding obtained in the previous chapter. This is the first main result of our thesis. The second main result is at the end of the chapter, and describes the geometry of the limiting harmonic maps. We show limits of minimal lagrangians in the bidisc are cores of $\mathbb{R}$-trees, thereby giving a geometric compactification of the Hitchin component of the $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ representation variety.

3.1 Metric structure of limits

3.1.1 Flat metrics as limits

In a simple scenario where the conformal structure of the minimal surface remains fixed, we can describe the asymptotic behavior of the induced metric. We consider the simplest case where $X_{1,n}$ (and consequently $X_{2,n}$) lie along a harmonic maps ray, that is the sequence of Hopf differentials of the projection map onto the first factor is given by $t_n\Phi$, where $\Phi \neq 0$ and $t_n \to \infty$.

Proposition 3.1. *Let $\sigma_n e_n$ be the induced metric where $\sigma_n = \sigma$ for all n, and the Hopf differentials of the harmonic maps $u_{1,n} : (S, \sigma) \to X_{1,n}$ are given by $t_n\Phi_0$, where Φ_0 is a unit-norm quadratic differential on (S, σ). Suppose $\mathcal{E}_n \to \infty$. Then everywhere away from the zeros of $|\Phi_0|$, one has*

$$\lim_{n\to\infty} \frac{\sigma_n e_n}{\mathcal{E}_n} = |\Phi_0|.$$

Proof. By construction, the Hopf differential of the harmonic map from (S, σ) to $X_{1,n}$ is given by $t_n\Phi_0$, where Φ_0 is a unit-norm quadratic differential. In a neighborhood away from any zero of Φ_0, consider then the horizontal foliation of $\Phi_n = t_n\Phi_0$. By the estimates on the geodesic curvature of its image [62], a horizontal arc of the foliation in this neighborhood will be mapped close to a geodesic in X_{1n} (we do not reproduce the techniques here, as we will do so later in a slightly modified setting). Using normal coordinates for the target adapted to this geodesic and estimates on stretching [61], we have that

$$(x, y) \mapsto (2t_n^{1/2}x, 0) + o(e^{-ct}),$$

where the constant c only depends upon the domain Riemann surface, and the distance from the zero of the quadratic differential. For the harmonic map from (S, σ) to $X_{2,n}$, its Hopf differential is given by $-t_n\Phi_0$, so that an arc of its horizontal foliation, which is an arc of the vertical foliation of $t_n\Phi_0$, gets mapped close to a geodesic, yielding

$$(x, y) \mapsto (0, 2t_n^{1/2} y) + o(e^{-ct}),$$

for some constant $c > 0$ that only depends on the Riemann surface structure on Σ. Hence, as a map from Σ to the 4-manifold $X_{1,n} \times X_{2,n}$ with the product metric, we have that the induced metric $\sigma_n e_n$ in this neighborhood has the form $(4t_n + o(e^{-ct}))dx^2 + 2o(e^{-ct})dxdy + (4t_n + o(e^{-ct}))dy^2$. Dividing by $4t_n$ and observing that for a high energy harmonic map, the total energy is comparable to twice the L^1-norm of the quadratic differential (Proposition 3.8), and taking the limit yields the conclusion.

□

Proposition 3.2. *Suppose $\sigma_n e_n$ is a sequence of induced metrics such that $\sigma_n \to \sigma \in \mathcal{T}(S)$ and $\mathcal{E}_n \to \infty$, then after passing to a subsequence, there exists a sequence t_n and a unit-norm quadratic differential Φ_0 on $[\sigma]$ so that*

$$\lim_{n \to \infty} \frac{\sigma_n e_n}{\mathcal{E}_n} \to |\Phi_0|.$$

Proof. The result follows from the compactness of unit-norm holomorphic quadratic differentials over a compact set in Teich(S) and the argument in the previous proposition.

□

As the previous results only show C^0 convergence in any neighborhood away from a zero of the quadratic differential, it is not quite so obvious we have convergence in the sense of length spectrum. The following technical proposition shows we actually do have convergence when the metrics are regarded as projectivized geodesic currents. With the length spectrum embedding (as given in Theorem 2.12), we now have sequences of points whose limits are the flat structures in the space of geodesic currents.

Proposition 3.3. *Let* $\sigma_n e_n$ *and* $\mathcal{E}_n$ *be in the same setting as above. Then as currents*

$$\frac{L_{\sigma_n e_n}}{\mathcal{E}_n^{1/2}} \to L_{|\Phi_0|}.$$

Proof. As the topology of geodesic currents is determined by the intersection number against closed curves, it suffices to show that given any closed, non-null homotopic curve class $[\gamma]$ and $\epsilon > 0$, there is an $N([\gamma], \epsilon)$ such that for $n > N$, one has that $|i(L_{\sigma_n e_n/\mathcal{E}_n}, \gamma) - i(L_{|\Phi_0|}, \gamma)| < \epsilon$. We choose a representative γ of $[\gamma]$ to be a $|\Phi_0|$-geodesic with length $L = i(L_{|\Phi_0|}, \gamma)$ with some fixed orientation. As the estimate in Proposition 3.2 does not hold near a zero z_i of $|\Phi_0|$, the first step is to construct open balls V_i of radius ϵ in the $|\Phi_0|$-metric about each zero z_i of Φ_0 (choosing ϵ sufficiently small) so that

(i) balls centered about distinct zeros do not intersect

(ii) if the curve γ enters one of the neighborhoods V_i, then the curve γ must intersect the zero z_i before γ exits V_i

(iii) $(1-\epsilon)\,C-(4g-2)\,\pi\,\epsilon > 0$, where C is the systolic length of the surface $(S, |\Phi_0|)$.

As Φ_0 is holomorphic, the zeros are isolated, so we can easily ensure (i) is satisfied. If the curve γ does not intersect z_i, then as γ is a closed curve, the distance from z_i to the curve γ in the $|\Phi_0|$-metric is bounded away from zero, guaranteeing condition (ii). Finally, condition (iii) follows as the systolic length C of $(S, |\Phi_0|)$ and the genus of surface are fixed.

As the complement of the union of the V_i's forms a compact set, by Proposition 3.2 we can find an N so that for $n > N$ the metrics $\sigma_n e_n/\mathcal{E}_n$ and $|\Phi_0|$ differ by at most ϵ. Now each time γ enters V_i, say at p, then hits the zero z_i and exits V_i for the first time thereafter, say at q, we may replace that segment of γ with a segment running along the boundary of V_i connecting p and q. Notice this does not change the homotopy class of γ. We make this alteration for each instance γ enters a V_i and denote the new curve by γ'. Observe that each time we make such an alteration, the length of the curve (in the $|\Phi_0|$ metric) increases by at most $K_i\epsilon$, where K_i is a constant depending only upon the $|\Phi_0|$ and the order of the zero z_i. In fact $K_i \leq (4g-2)\pi$. Hence the $|\Phi_0|$-length of γ' is bounded above by $L + \sum_{i=1}^{j} n_i K_i \epsilon$, where n_i is the number of times γ enters V_i. But as γ' now lies in the complement of the union of the V_i's, by Proposition 3.2, the length of γ' in the $\sigma_n e_n/\mathcal{E}_n$ metric is at most $(1+\epsilon)(L + \sum_{i=1}^{j} n_i K_i \epsilon)$. But the length of γ' in the $\sigma_n e_n/\mathcal{E}_n$ metric must be at least the length of the geodesic in its homotopy class, which has length $L'_n = i(L_{\sigma_n e_n/\mathcal{E}_n}, \gamma)$, hence

$$(1+\epsilon)(L + \sum_{i=1}^{j} n_i K_i \epsilon) \geq L'_n.$$

Distributing on the left hand side and subtracting both sides by L, yields

$$\begin{aligned}\sum_{i=1}^{j} n_i K_i \epsilon + \epsilon(L + \sum_{i=1}^{j} n_i K_i \epsilon) &\geq L'_n - L \\ &:= i(L_{\sigma_n e_n/\mathcal{E}_n}, \gamma) - i(L_{|\Phi_0|}, \gamma).\end{aligned}$$

Now if $L'_n - L \geq 0$, then we are done, for K_i is independent of ϵ and n_i decreases as ϵ does.

So consider the case where $L'_n - L < 0$, that is, $L > L'_n$. Consider the $\sigma_n e_n/\mathcal{E}_n$-geodesic $\widetilde{\gamma}_n$ in the homotopy class of γ, and again we give $\widetilde{\gamma}_n$ an orientation. Naturally $\widetilde{\gamma}_n$ can enter and exit the V_i's multiple times, but we remark that as the distance function on a NPC space from a convex set is itself convex, then each time the curve leaves V_i, it must pick up some topology before returning, that is, the part of the curve rel endpoints on lying on the boundary of V_i, the curve is not homotopic to a segment along the boundary of V_i.

However, now if $\widetilde{\gamma}$ enters and exits V_i say a total of r times, we consider the pairs of entry and exit points ordered accordingly as $p_1, q_1, ..., p_r, q_r$ using the chosen orientation. Now look at the segment of $\widetilde{\gamma}_n$ between p_s and p_{s+1}. If this is homotopic rel endpoints to a segment of the boundary of V_i, then we look at the segment of $\widetilde{\gamma}_n$ between p_s and p_{s+2} (using a cyclic ordering so that $r+1$ is identified with 1) and see if that segment is homotopic rel endpoints to a segment along the boundary of V_i. We repeat this until the segment of $\widetilde{\gamma}_n$ between p_s and $p_{s'}$ is not homotopic rel endpoints to the boundary of V_i. Then we repeat this process for p_s and p_{s-1} (again using a cylic ordering) until we find the segment of $\widetilde{\gamma}_n$ between p_s and $p_{s''}$ which is not homotopic rel endpoints to the boundary of V_i. Then we replace the segment of $\widetilde{\gamma}$ between $p_{s''+1}$ and $p_{s'-1}$ with a segment along the boundary of V_i connecting these two points. We repeat this for each i, so that when the curve leaves V_i, it picks up some topology before reentering V_i. Altering $\widetilde{\gamma}_n$ in this fashion yields a curve $\widetilde{\gamma}'_n$ lying outside of all the V_i's. Switching over the $|\Phi_0|$-metric yields the inequality,

$$(1+\epsilon)L'_n + \sum_{i=1}^{4g-4} m_i K_i \epsilon \geq L,$$

where m_i is the number of segments of the altered curve $\widetilde{\gamma}'_n$ lying on the boundary V_i and once again K_i is a constant depending solely on the order of the zero z_i. By assumption that $L > L'_n$, we have actually that

$$\begin{aligned} L'_n + \epsilon L + \sum_{i=1}^{4g-4} m_i K_i \epsilon &\geq L \\ \epsilon L + \sum_{i=1}^{4g-4} m_i K_i \epsilon &\geq L - L'_n. \end{aligned}$$

It suffices to show that m_i can be bounded independently of n. This follows from an estimate on the systolic length of the metric $\sigma e_n/\mathcal{E}_n$. Let C' denote the systolic length among all homotopically non-trivial curves which avoid the V_i's for the metric $|\Phi|$. Then $C' \geq C$. Then by Proposition 3.1, the systolic length among all homotopically non-trivial curves which avoid all the V_i's for the metric $\sigma e_n/\mathcal{E}_n$ is at least $(1-\epsilon)C'$.

If K denotes the largest constant among the K_i's, then one has that

$$\sum_{i=1}^{4g-4} m_i \leq \frac{L}{(1-\epsilon)C - K\epsilon},$$

for by construction we had m_i segments of $\widetilde{\gamma}'_n$ which are each not homotopic rel endpoints to the boundary of V_i, so that if we connect the endpoints of the segment with a segment along the boundary of V_i, we add at most $K\epsilon$ to the length of the segment. But we now have a closed curve not homotopic to the boundary of any of the V_i's, so the length of this closed curve is bigger than C'. This suffices for the proof.

□

The resulting flat metrics arising from unit-norm holomorphic quadratic differentials are distinct as Riemannian metrics from the induced metrics as the quadratic differential metrics have zero curvature away from the zeros, whereas the induced metrics have negative curvature everywhere (Proposition 2.5). In fact, the flat metrics are distinct as geodesic currents, as work of Frazier [20] shows the marked length spectrum distinguishes nonpositively curved Euclidean metrics from the negatively curved Riemannian metrics.

3.1.2 Measured laminations as limits

However, not all limits of induced metrics are given by flat metrics. One can also obtain measured laminations. This is most readily seen in the setting where one takes a hyperbolic metric and looks at the minimal lagrangian to itself. The induced metric of the minimal surface is then twice the hyperbolic metric. We thus have a copy of Teichmüller space inside the space of induced metrics inside the space of projectivized currents. From Bonahon [3], we know we must have projectivized measured laminations in our compactification of the induced metrics. However, there are more ways to obtain measured laminations than by degenerating only the induced metrics which are scalar multiples of hyperbolic metrics, as the following proposition shows.

Proposition 3.4. *Suppose $L_{\sigma_n e_n}$ leaves all compact sets, but that the sequence $\mathcal{E}_n$ of total energies is bounded, then in* $\mathrm{PCurr}(S)$*, we have* $[L_{\sigma_n e_n}] \to [\lambda] \in \mathrm{PMF}(S)$*. Furthermore, if* $[L_{\sigma_n}] \to [\lambda']$ *in the Thurston compactification, then* $i(\lambda, \lambda') = 0$*,* *where* $\lambda \in [\lambda]$ *and* $\lambda' \in [\lambda']$.

Proof. By the compactness of PCurr(S), any sequence $[L_{\sigma_n e_n}]$ subconverges to $[\lambda] \in$ PCurr(S). Hence, there is a sequence of positive real numbers so that $t_n L_{\sigma_n e_n} \to \lambda \in$ Curr(S). We claim $t_n \to 0$.

Consider a finite set of curves $\gamma_1, \gamma_2, ..., \gamma_k$ which fill the surface S. Then the current $\gamma_1 + \gamma_2 + ... + \gamma_k$ is a *binding current*, that is to say, it has positive intersection number with any non-zero geodesic current.

As $L_{\sigma_n e_n}$ leaves all compact sets in Curr(S), then

$$\lim_{n\to\infty} i(L_{\sigma_n e_n}, \gamma_1 + ... + \gamma_k) \to \infty,$$

so by continuity of the intersection form, one has

$$\lim_{n\to\infty} t_n i(L_{\sigma_n e_n}, \gamma_1 + ... + \gamma_k) = i(\lambda, \gamma_1 + ... + \gamma_k).$$

But the intersection number on the right hand side is finite, hence $t_n \to 0$. From Proposition 1.5, one has $i(L_{\sigma_n e_n}, L_{\sigma_n e_n}) = \pi/2$ Area$(S, \sigma_n e_n)$, which in this case is $\frac{\pi}{2}\mathcal{E}_n$. Then

$$\begin{aligned} i(\lambda, \lambda) &= \lim_{n\to\infty} i(t_n L_{\sigma_n e_n}, t_n L_{\sigma_n e_n}) \\ &= \lim_{n\to\infty} t_n^2 \frac{\pi}{2} \mathcal{E}_n = 0, \end{aligned}$$

where the last equality follows from the boundedness of total energy, hence $\lambda \in$ MF(S). Now if $[L_{\sigma_n}] \to [\lambda']$, then there is a sequence $t'_n \to 0$ such that $t'_n L_{\sigma_n} \to \lambda'$.

Then

$$\begin{aligned} i(\lambda,\lambda') &= \lim_{n\to\infty} i(t_n L_{\sigma_n e_n}, t'_n L_{\sigma_n}) \\ &\leq \lim_{n\to\infty} t_n t'_n i(L_{\sigma_n e_n}, L_{\sigma_n e_n}) \\ &= \lim_{n\to\infty} t_n t'_n \mathcal{E}_n = 0. \end{aligned}$$

where the inequality follows from $\sigma_n \leq \sigma_n e_n$ as metrics, and the last equality by the boundedness of the sequence of total energy $\mathcal{E}_n$ along with the sequences t_n, t'_n tending towards zero. □

3.1.3 Mixed structures as limits

As some of the possible limits are the singular flat metrics arising from a holomorphic quadratic differential, the closure of the space of induced metrics on the minimal surface must include mixed structures, as these arise as limits of singular flat metrics [16]. The main theorem asserts these are precisely all the possible limits of the degenerating minimal surfaces.

Theorem 3.5. *Let $\sigma_n e_n$ be a sequence of induced metrics such that either σ_n leaves all compact sets in $\mathcal{T}(S)$ or $\mathcal{E}_n \to \infty$, then there exists a sequence $t_n \to 0$ so that up to a subsequence $t_n L_{\sigma_n e_n} \to \eta = (S', q, \lambda) \in \mathrm{Mix}(S) \subset \mathrm{Curr}(S)$. Furthermore, given any $\eta \in \mathrm{Mix}(S)$, there exists a sequence of induced metrics $\sigma_n e_n$ and a sequence of constants $t_n \to 0$, so that $t_n L_{\sigma_n e_n} \to \eta$. Hence, the closure of the space of induced metrics in the space of projectivized currents is $\overline{\mathrm{Ind}(S)} = \mathrm{Ind}(S) \sqcup \mathrm{PMix}(S)$.*

The proof of the main theorem will follow from a series of intermediate results, and will be at the end of the section. The strategy is to show that if the sequence of currents coming from the induced metrics is not converging projectively to a measured lamination, then scaling the induced metrics to have total area 1 is enough to ensure convergence in length spectrum. To each normalized induced metric, we produce a quadratic differential metric in the same conformal class as the induced metric, which will serve as a lower bound. Convergence of the quadratic differential metric to a mixed structure will yield a decomposition of the surface into a flat part and a laminar part. On each flat part, we will prove the conformal factor between the normalized induced metric and the quadratic differential converges to 1 uniformly (away from finitely many points). An area argument will show the complement is laminar.

The following proposition allows us to analyze sequences of induced metrics which are not converging to projectivized measured laminations. If the sequence of induced metrics is not converging to a projectivized measured lamination, we may scale the current associated to the induced metric by the square root of its area (which is also the total energy). We remark that in the case where the limiting geodesic current is not a measured lamination, then scaling the induced metrics by total energy of the associated harmonic map is strong enough to ensure length-spectrum convergence, yet delicate enough to ensure the limiting length spectrum is not identically zero. This should be compared to the situation in [62] and [14] where one always scales the metric by the total energy.

Proposition 3.6. *Suppose the conformal class of the minimal surface leaves all compact sets in $\mathcal{T}(S)$ and the sequence of total energies is unbounded, that is $\mathcal{E}_n \to \infty$.*

Then up to a subsequence, there exists a sequence $c_n \to 0$ *and a geodesic current* μ *such that* $c_n L_{\sigma_n e_n} \to \mu$*. If* μ *is a measured lamination, then* $c_n = o(\mathcal{E}_n^{-1/2})$*. If* μ *is not a measured lamination, then* $c_n \asymp \mathcal{E}_n^{-1/2}$*.*

Proof. By Theorem 2.12, one has an embedding of the space of induced metrics into the space of projectivized geodesic currents, which is compact. Taking the closure implies the first result. If $[\mu]$ is the limiting projective geodesic current, then one can choose a fixed representative; call it μ.

If μ is a measured lamination, then dividing the current $L_{\sigma_n e_n}$ by $\mathcal{E}_n^{1/2}$ normalizes the current to have self-intersection number 1. Then as the measured laminations have self-intersection 0, the second result follows.

Suppose then μ is not a measured lamination. Then its self-intersection number is positive and finite. But

$$\begin{aligned}
i(\mu, \mu) &= \lim_{n\to\infty} i(c_n L_{\sigma_n e_n}, c_n L_{\sigma_n e_n}) \\
&= \lim_{n\to\infty} c_n^2\, i(L_{\sigma_n e_n}, L_{\sigma_n e_n}) \\
&= \lim_{n\to\infty} c_n^2\, \frac{\pi}{2}\, \mathrm{Area}(S, \sigma_n e_n) \\
&= \lim_{n\to\infty} c_n^2\, \frac{\pi}{2} \int_S \sigma_n e_n\, dz_n \wedge d\overline{z}_n \\
&= \lim_{n\to\infty} c_n^2\, \frac{\pi}{2}\, \mathcal{E}_n,
\end{aligned}$$

so that $0 < \lim_{n\to\infty} c_n^2\, \mathcal{E}_n < \infty$, that is $c_n \asymp \mathcal{E}_n^{-1/2}$, as desired.

□

With the normalization of dividing the current $L_{\sigma_n e_n}$ by $\mathcal{E}_n^{1/2}$, the self-intersection of the current will be $\pi/2$, that is to say we have scaled the induced metric to have total area 1.

The following proposition shows the relation of the induced metric to the corresponding Hopf differential metric.

Proposition 3.7. *Away from the zeros of* Φ*, one has the following identity*

$$\sigma e = |\Phi| \left(\frac{1}{|\nu|} + |\nu| \right).$$

Consequently,

$$\sigma_n e_n \geq 2|\Phi_n|.$$

Proof. This result follows immediately by manipulation of the formulae involving $\mathcal{H}$ and $\mathcal{L}$. One has

$$\begin{aligned} \sigma^2 e^2 &= \sigma^2(\mathcal{H}^2 + 2\mathcal{H}\mathcal{L} + \mathcal{L}^2) \\ &= \sigma^2 \mathcal{H}\mathcal{L} \left(\frac{\mathcal{H}}{\mathcal{L}} + 2 + \frac{\mathcal{L}}{\mathcal{H}} \right) \\ &= |\Phi|^2 \left(\frac{1}{|\nu|}^2 + 2 + |\nu|^2 \right). \end{aligned}$$

Taking a square root on both sides yields the result. □

For a given sequence $\sigma_n e_n$ we consider the associated smooth (away from the zeros of the quadratic differential) function $f_n = (\frac{1}{|\nu_n|} + |\nu_n|)$. To each n, there is only one such f_n by Lemma 2.8.

The following proposition due to Wolf allows us to pass freely between the L^1-norm of a Hopf differential and the total energy of the corresponding harmonic map. The original proof was for a fixed Riemann surface as the domain, but the argument holds when the domain is allowed to change. For the ease of the reader, we have included the adapted proof.

Proposition 3.8 (Wolf [61], Lemma 3.2)**.** *For any Riemann surface* (S, J) *and hyperbolic surface* (S, σ)*, if* $\mathrm{id} : (S, J) \to (S, \sigma)$ *is a harmonic map with Hopf differential* Φ *and total energy* $\mathcal{E}$*, then*

$$\mathcal{E} + 2\pi\chi(s) \leq 2||\Phi|| \leq \mathcal{E} - 2\pi\chi(S).$$

Proof. As $\mathcal{H} - \mathcal{L} = \mathcal{J}$ and $\int \mathcal{J}\sigma dz d\overline{z} = -2\pi\chi$, we have

$$\int \mathcal{H}\,\sigma\, dz d\overline{z} + 2\pi\chi = \int \mathcal{L}\,\sigma\, dz d\overline{z} = \int \Phi\,\nu\, dz d\overline{z},$$

as the integrands agree. But, recalling that $|\nu| < 1$, we have

$$\begin{aligned} \int \Phi\,\nu\, dz d\overline{z} &\leq \int |\Phi|\, dz d\overline{z} \\ &= \int \mathcal{H}\,|\nu|\,\sigma\, dz d\overline{z} \\ &\leq \int \mathcal{H}\,\sigma\, dz d\overline{z} = \int \mathcal{L}\,\sigma\, dz d\overline{z} - 2\pi\chi. \end{aligned}$$

Adding the first integral and the last integral yields

$$\int e\sigma\, dz d\overline{z} + 2\pi\chi \leq 2\int |\Phi|\, dz d\overline{z} \leq \int e\,\sigma\, dz d\overline{z} - 2\pi\chi,$$

proving the proposition. □

Corollary 3.9. *If the sequence* $\Phi_{0,n}$ *of unit-norm quadratic differential metrics converges projectively to a measured lamination, then so does the associated sequence* $L_{\sigma_n e_n}/\mathcal{E}_n^{1/2}$ *of geodesic currents.*

Proof. Suppose $L_{|\Phi_{0,n}|} \to [\lambda]$ in the space of projectivized currents. As $i(L_{|\Phi_{0,n}|}, L_{|\Phi_{0,n}|}) = \pi/2$, while $i(\lambda, \lambda) = 0$, then there exists a sequence $t_n \to 0$, so that the length spectrum of $t_n|\Phi_{0,n}|$ converges to that of some $\lambda \in [\lambda]$. This is to say, there is a curve class $[\gamma]$, for which the length of its geodesic representative against the metric $|\Phi_{0,n}|$ is unbounded, so by Propositions 3.7 and 3.8, the sequence of lengths of the $[\gamma]$-geodesic against the metrics $\sigma_n e_n/\mathcal{E}_n$ is unbounded. Hence there is a sequence $s_n \to 0$ so that $s_n L_{\sigma_n e_n}/\mathcal{E}_n^{1/2}$ converges to a current μ. But as the self-intersection of $L_{\sigma_n e_n}/\mathcal{E}_n$ is exactly $\pi/2$, the intersection of μ with itself is zero, from which the result follows. □

The previous corollary allows us to exclude the case where the sequence of flat metrics tends towards a projectivized measured lamination, for in that case, we have that the sequence of induced metrics also tends towards a projectivized measured lamination. Hence, we need only consider the case where the sequence of flat metrics converges to a non-trivial mixed structure, say η. The data of η gives us a subsurface S' for which the restriction of η is a flat metric arising from a quadratic differential. Here we consider S' up to isotopy.

The remainder of the section is devoted towards showing that if the sequence of unit-norm quadratic differential metrics converges to a mixed structure that is not entirely laminar, then so does the sequence of unit-area induced metrics. This will then complete the proof of Theorem 3.5.

We begin by record the following useful bound due to Minsky, for the function $\mathcal{G} = \log(1/|\nu|)$.

Proposition 3.10 (Minsky [45], Lemma 3.2)**.** *Let $p \in S$ be a point with a neighborhood U such that U contains no zeros of Φ and in the $|\Phi|$-metric is a round disk of radius r centered on p. Then there is a bound*

$$\mathcal{G}(p) \leq \sinh^{-1}\left(\frac{|\chi(S)|}{r^2}\right).$$

Proof. The PDE $\Delta\mathcal{G} = 2\mathcal{J} > 0$ shows that $\mathcal{G}$ is subharmonic in U. It suffices therefore to bound the average of $\mathcal{G}$ on U in the $|\Phi|$-metric. Some algebra yields

$$\sinh \mathcal{G} = \frac{1}{2}\frac{\sigma}{|\Phi|}\mathcal{J}.$$

Using the concavity of $\sinh^{-1}$ on the positive real axis, we obtain

$$\begin{aligned}
\mathcal{G}(p) &\leq |\Phi|\text{-Avg}_U(\mathcal{G}) && \text{by subharmonicity of } \mathcal{G} \\
&= |\Phi|\text{-Avg}_U\left(\sinh^{-1}\frac{1}{2}\frac{\sigma}{|\Phi|}\right) \\
&\leq \sinh^{-1}\left(|\Phi|\text{-Avg}_U\left(\frac{1}{2}\frac{\sigma}{|\Phi|}\right)\right) && \text{by concavity of } \sinh^{-1} \\
&= \sinh^{-1}\left(\frac{1}{2\pi r^2}\int_U \frac{\sigma}{|\Phi|}\,\mathcal{J}\,dA(|\Phi|)\right) \\
&\leq \sinh^{-1}\left(\frac{|\chi(S)|}{r^2}\right) && \text{by Gauss-Bonnet.}
\end{aligned}$$

□

As we are in the setting where the sequence $L_{\Phi_{0,n}}$ of currents coming from unit-area holomorphic quadratic differential metrics converges to a non-trivial mixed structure $\eta = (S', \Phi_\infty, \lambda)$, we have that the restriction of the metric $|\Phi_{0,n}|$ to S' converges to the metric $|\Phi_\infty|$. On this systole positive collection S' of subsurfaces, we have the following proposition.

Proposition 3.11. *Given $\epsilon, \epsilon' > 0$, there exists $N = N(\epsilon, \epsilon')$ such that for $n > N$*

$$m_{|\Phi_{0,n}|}(\{p \in S : \left(\frac{1}{|\nu_n|} + |\nu_n|\right)(p) \geq 2 + \epsilon'\}) < \epsilon.$$

Consequently the limiting function $\frac{1}{|\nu|} + |\nu|$ is 2 almost everywhere with respect to the $|\Phi_\infty|$-metric.

Proof. By Proposition 3.7, one has the equality

$$\begin{aligned}\frac{\sigma_n e_n}{\mathcal{E}_n} &= \frac{|\Phi_n|}{\mathcal{E}_n}\left(\frac{1}{|\nu_n|}+|\nu_n|\right)\\ &= \left(\frac{||\Phi_n||}{\mathcal{E}_n}\right)\frac{|\Phi_n|}{||\Phi_n||}\left(\frac{1}{|\nu_n|}+|\nu_n|\right).\end{aligned}$$

Defining

$$\frac{1-c_n}{2} := \left(\frac{||\Phi_n||}{\mathcal{E}_n}\right),$$

one has $c_n \to 0$ by virtue of Proposition 3.8. Observe that $c_n \geq 0$, as the function $\frac{1}{|\nu_n|}+|\nu_n| \geq 2$, the area of $|\Phi_{0,n}| = \frac{|\Phi_n|}{||\Phi_n||}$ is 1 and the area of the scaled metric $\frac{\sigma_n e_n}{\mathcal{E}_n}$ is also 1. If m_n then denotes the $|\Phi_{0,n}|$-measure of the set of of points for which the function $\frac{1}{|\nu_n|}+|\nu_n|$ is at least $2+\epsilon'$, then one has

$$\begin{aligned}&\int_{\{p:(\frac{1}{|\nu_n|}+|\nu_n|)(p)\geq 2+\epsilon'\}}\left(\frac{1}{|\nu_n|}+|\nu_n|\right)\left(\frac{1-c_n}{2}\right)\, dA(|\Phi_{0,n}|)\\ &+\int_{\{p:(\frac{1}{|\nu_n|}+|\nu_n|)(p)< 2+\epsilon'\}}\left(\frac{1}{|\nu_n|}+|\nu_n|\right)\left(\frac{1-c_n}{2}\right)\, dA(|\Phi_{0,n}|) = \int dA(\frac{\sigma_n e_n}{\mathcal{E}_n}) = 1.\end{aligned}$$

The integrand in the first integral is at least $(2+\epsilon')\left(\frac{1-c_n}{2}\right)$, whereas the second integrand is at least $2\left(\frac{1-c_n}{2}\right)$. Multiplying these lower bounds with the measures of their respective sets yields

$$(2+\epsilon')\left(\frac{1-c_n}{2}\right)m_n + 2\left(\frac{1-c_n}{2}\right)(1-m_n) \leq 1.$$

Some basic algebraic manipulation gives

$$\begin{aligned}m_n\left((2+\epsilon')\left(\frac{1-c_n}{2}\right)-2\left(\frac{1-c_n}{2}\right)\right) &\leq c_n\\ m_n\left(\frac{1-c_n}{2}\right)\epsilon' &\leq c_n\\ m_n &\leq \frac{2c_n}{(1-c_n)(\epsilon')},\end{aligned}$$

and as ϵ' is now fixed, one may find a sufficiently large N to guarantee $m_n < \epsilon$. As the metric $|\Phi_\infty|$ has finite total area, convergence in measure of the sequence of functions $\frac{1}{|\nu_n|} + |\nu_n|$ to the constant function 2, implies that up to a subsequence, one has convergence to the constant function 2 almost everywhere. □

Sets of measure zero can be rather problematic if we wish to say something about length of curves. The following proposition shows that we actually have convergence off the zeros and poles of $|\Phi_\infty|$.

Proposition 3.12. *Suppose $\mathcal{E}_n \to \infty$. Then up to a subsequence $\left(\frac{1}{|\nu_n|} + |\nu_n|\right) \to 2$ everywhere on S' except at the zeros and poles of $|\Phi_\infty|$.*

Proof. Observe that the function $\frac{1}{|\nu_n|} + |\nu_n|$ is not defined at the zeros of $|\Phi_n|$, but is well-defined everywhere else. Moreover, the auxiliary function $\mathcal{G} = \log \frac{1}{|\nu|}$ satisfies the partial differential equation

$$\Delta \log \frac{1}{|\nu_n|} = 2\mathcal{J}_n > 0,$$

so that the function $\mathcal{G}$ and hence $\frac{1}{|\nu_n|} + |\nu_n|$ never attains an interior maximum on the complement of the zeros. It follows that $\frac{1}{|\nu_n|} + |\nu_n|$ is only unbounded in a neighborhood of a zero of a corresponding quadratic differential Φ_n. The sequence of flat metrics $|\Phi_{0,n}|$ on S' converges geometrically to $|\Phi_\infty|$, and so the zeros of $|\Phi_n|$ on S' will converge to the zeros of $|\Phi_\infty|$. For any $\epsilon > 0$, consider balls of radius 3ϵ about each zero of $|\Phi_\infty|$, choosing ϵ sufficiently small, so that balls about distinct zeros do not intersect. Call this collection B. Then for large n, balls of radius ϵ in the $|\Phi_{0,n}|$ metric about the zeros of $|\Phi_n|$ will be contained in B. For each boundary component of S', which in the geometric limit is collapsed to a puncture, choose a geodesic curve with respect to the $|\Phi_\infty|$-metric, homotopic to the puncture and enclosing the puncture, of length $l_\epsilon > 3\epsilon$ so that the $|\Phi_\infty|$-distance of each point of the curve to the puncture is at least 3ϵ, possibly choosing a smaller ϵ until such a configuration is possible. This gives an annulus for each boundary component of S'. Call the collection of these annuli A.

For any point in the complement of both A and B, for large n, the injectivity radius with respect to the $|\Phi_{0,n}|$-metric is at least ϵ and the distance to any of the zeros is at least ϵ. Moreover, each point p in the region satisfies the property that any $q \in B_{\epsilon/2}(p)$ has injectivity radius at least $\epsilon/2$ and distance at least $\epsilon/2$ to any zero or the boundary of the cylindrical region. Hence, by Proposition 3.10, the value of $\log(1/|\nu_n|)$ is at most $M_{\epsilon/2}$, where the constant no longer depends on n, once n is chosen sufficiently large. As the function $\log(1/|\nu_n|)$ is subharmonic, by the mean-value property, one has at any point p in this set

$$\begin{aligned} \log(1/|\nu_n|)(p) &\leq \int_{B_{\epsilon/2}(p)} \log(1/|\nu_n|)\ dA_{|\Phi_{0,n}|} \\ &\leq \left(|\Phi_{0,n}|\text{-Area}(B_{\epsilon/2}(p)\right) \epsilon' + M_{\epsilon/2}\epsilon'', \end{aligned}$$

for n large enough so that $\log(1/|\nu_n|) < \epsilon'$ outside a set of measure at most ϵ'' by Proposition 3.11. As the choice of ϵ was arbitrary, the conclusion follows.

□

These collection of propositions prove the following result:

Theorem 3.13. *Suppose $L_{|\Phi_{0,n}|}$ converges to a mixed structure η, which is not en-*

tirely a measured lamination. Then the corresponding metrics $\sigma_n e_n/\mathcal{E}_n$ as $\mathcal{E}_n \to \infty$, restricted to S' converges geometrically to $|\Phi_\infty|$.

Proof. Defining A and B as in the previous proof, on the region $S' \backslash (A \cup B)$, Proposition 3.12 guarantees that we have uniform bounds on the sequence of functions $1/|\nu_n| + |\nu_n|$ whose limit was the constant function 2. Hence by Arzelá-Ascoli, up to a subsequence, we have uniform convergence on this region. Hence, by the same argument as that of Proposition 3.3, the length spectrum of the scaled induced metric on this domain converges to the limiting length spectrum of the sequence $|\Phi_{0,n}|$, which is $|\Phi_\infty|$. □

Proof of Theorem 3.5. Recall that to any flat metric arising from a holomorphic quadratic differential, one can find a sequence of induced metrics so that the chosen flat metric is the limit in the space of geodesic currents (Proposition 3.2). Hence by Theorem 1.10, any mixed structure η can be obtained by a sequence $L_{\sigma_n e_n}$ of currents coming from the induced metrics. On the other hand, to any sequence of induced metrics leaving all compact sets, then either it converges projectively to a measured lamination or it does not. If it does not, then the corresponding sequence of normalized Hopf differential metrics must converge to a mixed structure which is not purely laminar. The previous theorem thus ensures there is a nonempty collection of incompressible subsurfaces, S', on which the limiting current is a flat metric. But on the complement of S', the current μ restricts to a measured lamination (as on this complement the areas of the metric tend to zero), the proof of Theorem 3.5 is complete. □

3.1.4 Dimension of the boundary

We end this section with a remark about the compactification of the induced metrics. Recall the dimension of the space of induced metrics (being homeomorphic to $\mathcal{Q}_g/\mathbb{S}^1$) was $12g-13$. The dimension of the singular flat metrics can be readily seen to be of dimension $12g-14$. The mixed structures are stratified by the subsurfaces for which the mixed structure is a flat metric. A subsurface of lower complexity yields fewer free parameters than the whole surface S in the choice of a flat structure, and the extra choices one gains for a measured lamination on the complementary subsurface is strictly less than the loss of choice for the flat structure. Hence the boundary of the compactification of the induced metrics via projectivized geodesic currents is of codimension 1.

3.2 Geometry of the limits

In this section, we wish to relate the mixed structures with cores of $\mathbb{R}$-trees arising from laminations. To this end, we elucidate the relation between the mixed structure and the pair of projective measured laminations obtained from the pair of degenerating hyperbolic surfaces.

3.2.1 $\mathbb{R}$-trees

Here we recall some basic facts about $\mathbb{R}$-trees. An $\mathbb{R}$-tree T is a metric space for which any two points are connected by a unique topological arc, and such that the arc is a geodesic. Equivalently, if (X,d) is a metric space, for any pair of points $x,y \in X$, define the segment $[x, y] = \{z \in X \,|\, d(x, y) = d(x, z) + d(z, y)\}$. Then an $\mathbb{R}$-tree is a real non-empty metric space (T, d) satisfying the following:

(i) for all $x, y \in T$, the segment $[x, y]$ is isometric to a segment in $\mathbb{R}$.

(ii) the intersection of two segments with an endpoint in common is a segment

(iii) the union of two segments of T whose intersection is a single point which is an endpoint of each is itself a segment.

We say that a group Γ acts on T *by isometry* if there is a group homomorphism $\theta : \Gamma \to \mathrm{Isom}(T)$. The action is from the left. An action is said to be *small* if the stabilizer of each arc does not contain a free group of rank 2. An action is said to be *minimal* if no proper subtree is invariant under Γ.

A particularly important class of $\mathbb{R}$-trees comes from the leaf space of a lift of a measured foliation on a closed surface to its universal cover. Any measured foliation $\mathcal{F}$ on a closed surface of genus $g \geq 2$ may be lifted to a $\pi_1 S$-equivariant measured foliation on its universal cover. The leaf space can be made into a metric space, by letting the distance be induced from the intersection number. Notice this is an $\mathbb{R}$-tree with a $\Gamma = \pi_1 S$ action by isometries. Naturally, not all $\mathbb{R}$-trees with a $\pi_1 S$ action arise from this construction. A theorem of Skora [55] shows that an $\mathbb{R}$-tree with a $\pi_1 S$-action comes from a measured foliation if and only if the action is small and minimal (see [21] for a proof in the current setting of harmonic maps). Alternatively, one may start with a measured lamination (λ, μ) on S and lift it to a measured lamination $(\widetilde{\lambda}, \mu)$ on the universal cover. Then an $\mathbb{R}$-tree may be formed by taking the connected components of $\widetilde{S} \backslash \widetilde{\lambda}$ with edges between two vertices if the two components were adjacent (separated by a geodesic), and then metrically completing the distance induced by the intersection number. The $\mathbb{R}$-tree comes equipped with a $\pi_1 S$-action, and is $\pi_1 S$-equivariantly isometric to the $\mathbb{R}$-tree constructed from the corresponding measured foliation. In what follows, we will deal exclusively with $\mathbb{R}$-trees with a $\pi_1 S$-action coming from the leaf space of the lift of a measured foliation.

3.2.2 Relation of flat metrics to $\mathbb{R}$-trees

We obtain a classification of the flat parts of the mixed structure arising from the data of the limits of the sequences $X_{1,n}$ and $X_{2,n}$. Let S' be a connected subsurface for which the limiting mixed structure η is a flat metric. For each n, denote by S'_n the subsurface isotopic to S' such that the boundary components are geodesics with respect to the induced metric $\sigma_n e_n/\mathcal{E}_n$. Let $X'_{1,n}$ denote the restriction of the hyperbolic metric $X_{1,n}$ to the subsurface of S, in the same isotopy class of S', but which has geodesic boundary with respect to the restricted hyperbolic metric. Then let $u'_{i,n}$ denote the restriction to S'_n of the harmonic map $u_{i,n} : (S, \sigma_n e_n) \to X_{i,n}$.

Theorem 3.14. *Consider a connected component of S'. The sequence of harmonic maps $u'_{1,n} : (S'_n, \sigma_n e_n/\mathcal{E}_n) \to X_{1,n}/2\mathcal{E}_n$ converges to a $\pi_1(S')$-equivariant harmonic map $u' : (S', |\Phi_\infty|) \to T_1$, where T_1 is the $\mathbb{R}$-tree dual to $\lambda_1 = \lim_{n\to\infty} X_{1,n}/2\mathcal{E}_n$. The Hopf differential is given by Φ_∞. Likewise the same holds for λ_2 and $-\Phi_\infty$. Hence, the laminations are the vertical and horizontal foliations of Φ_∞.*

Proof. We begin by showing that λ_1 is a well-defined measured lamination in the projective class of $[\lambda_1]$, which is the limit on the Thurston boundary of the sequence

$X_{1,n}$. This will follow from standard estimates on stretching and geodesic curvature of an arc of the horizontal foliation which avoids the zeros. This will be an adaptation of the argument employed in [62], for the case where the domain conformal structure is fixed and the Hopf differentials lie along a ray.

We first show boundedness of the Jacobian. For any neighborhood U of the surface which avoids a zero of $\Phi_{0,n}$ one has the usual PDE

$$\Delta_{\sigma_n} \log \frac{1}{|\nu_n|^2} = 4\mathcal{J}_n > 0, \tag{3.1}$$

and consequently,

$$\Delta_{\sigma_n} ||\Phi_n|| \log \frac{1}{|\nu_n|^2} = 4||\Phi_n||\mathcal{J}_n > 0. \tag{3.2}$$

Using the conformal invariance of harmonic maps, we replace the metric σ_n on the neighborhood U with a metric σ_n' in the same conformal class as σ_n, but one which is flat on U. Subharmonicity of the function $||\Phi_n|| \log 1/|\nu_n|^2$ yields

$$||\Phi_n|| \log \frac{1}{|\nu_n|^2}(p) \leq \frac{1}{\pi R^2} \int_{B_R(p)} ||\Phi_n|| \log \frac{1}{|\nu_n|^2} \; dA(\sigma_n') \tag{3.3}$$

on a ball of σ_n' radius R contained in U. Some algebra yields

$$\mathcal{J}_n(p) \frac{||\Phi_n|| \log \frac{1}{|\nu_n|^2}(p)}{\mathcal{J}_n(p)} \leq \frac{1}{\pi R^2} \int_{B_R(p)} \frac{\mathcal{J}_n}{\mathcal{J}_n} ||\Phi_n|| \log \frac{1}{|\nu_n|^2} \; dA(\sigma_n'), \tag{3.4}$$

and hence

$$\mathcal{J}_n(p) \leq \frac{\mathcal{J}_n(p)}{||\Phi_n|| \log |\nu_n|^{-2}} \left(\sup_{q \in B_R(p)} \frac{||\Phi_n|| \log |\nu_n|^{-2}(q)}{\mathcal{J}_n(q)} \right) \frac{1}{\pi R^2} \int_{B_R(p)} \mathcal{J}_n \; dA(\sigma_n'). \tag{3.5}$$

But one has that

$$\frac{\mathcal{J}_n}{||\Phi_n|| \log |\nu_n|^{-2}} = \frac{|\Phi_{0,n}|}{\sigma_n |\nu_n|} \frac{(1 - |\nu_n|^2)}{\log |\nu_n|^{-2}}, \tag{3.6}$$

so that in applying Proposition 3.12 to the expression (6.6), one obtains that (6.5) may be rewritten as

$$\mathcal{J}_n(p) \leq c_n \int_{B_R(p)} \mathcal{J}_n \; dA(\sigma_n'), \tag{3.7}$$

where c_n will depend on the metric $|\Phi_{0,n}|, |\nu_n|, R$ and σ_n. But on the neighborhood U, we know for sufficiently large n, we have that $|\Phi_n| \to |\Phi_\infty|$, and $|\nu_n| \to 1$ and $\sigma_n \to \sigma_\infty$, where σ_∞ is the uniformizing metric of Φ_∞. Hence c_n remains bounded on U. But finally,

$$\int_{B_R(p)} \mathcal{J}_n \; dA(\sigma_n') = \int_{B_R(p)} \frac{\sigma_n'}{\sigma_n} \mathcal{J}_n \; dA(\sigma_n) \leq \sup_U \frac{\sigma_n'}{\sigma_n} \int_M \mathcal{J}_n \; dA(\sigma_n) \leq -2\pi\chi(S) c_n', \tag{3.8}$$

where here c_n' will only depend upon the injectivity radius of the metric σ_n on the neighborhood U, which for large n will be close to the injectivity radius of σ_∞.

From (6.7), (6.8) and the PDE in (6.1), one obtains by elliptic regularity (see [23], Problem 4.8a) that $|\nu_n| \to 1$ in $C^{1,\alpha}(U)$, where U does not contain a zero or pole of Φ_∞.

In the natural coordinates of the quadratic differential, the hyperbolic metric $g_{1,n}$ is given by $(\sigma_n e_n + 2||\Phi_n||) d\zeta_n^2 + (\sigma_n e_n - 2||\Phi_n||) d\eta_n^2$.

Recall that the geodesic curvature of an arc of the horizontal foliation of $\Phi_{0,n}$ in the natural coordinates for $\Phi_{0,n} = d\zeta_n^2 = d\xi_n^2 + d\eta_n^2$ is given by the equation

$$\kappa(\gamma)_{\eta=constant} = -\frac{1}{2g_{11}\sqrt{g_{22}}} \frac{\partial g_{11}}{\partial \eta_n}, \tag{3.9}$$

so that for γ an arc of the horizontal foliation of $\Phi_{0,n}$ avoiding the zeros, one has

$$\kappa(\gamma)_{\eta=\text{constant}} = -\frac{1}{2(\sigma_n e_n + 2||\Phi_n||)(\sigma_n e_n - 2||\Phi_n||)^{1/2}} \frac{\partial}{\partial \eta_n}(\sigma_n e_n + 2||\Phi_n||) \tag{3.10}$$

$$= -\frac{1}{2\mathcal{J}_n(\sigma_n e_n + 2||\Phi_n||)^{1/2}} \frac{\partial}{\partial \eta_n} \sigma_n e_n. \tag{3.11}$$

But simple algebra yields that $\sigma_n e_n = ||\Phi_n|||\Phi_{0,n}|(|\nu_n|^{-1} + |\nu_n|)$, so that in the natural coordinates as $|\Phi_{0,n}| \equiv 1$, one actually has $\sigma_n e_n = ||\Phi_n||(|\nu_n|^{-1} + |\nu_n|)$. Hence

$$\kappa(\gamma) = \frac{1}{2}||\Phi_n||(1 - |\nu_n|^2)\mathcal{J}_n^{-1}|\nu_n|^{-2}(\sigma_n e_n + 2||\Phi_n||)^{-1/2} \frac{\partial}{\partial \eta_n}|\nu_n| \tag{3.12}$$

$$= \frac{1}{2}||\Phi_n||\mathcal{H}_n^{-1}|\nu_n|^{-2}(\sigma_n e_n + 2||\Phi_n||)^{-1/2} \frac{\partial}{\partial \eta_n}|\nu_n|, \tag{3.13}$$

as $\mathcal{J}_n = \mathcal{H}_n(1 - |\nu_n|^2)$. As $||\Phi_n||\mathcal{H}_n^{-1} = |\nu_n|/|\Phi_{0,n}|$, rewriting (6.13) gives

$$\kappa(\gamma) = \frac{1}{2} \frac{1}{(|\Phi_{0,n}| \cdot |\nu_n|)(\sigma e_n + 2||\Phi_n||)^{1/2}} \cdot \frac{\partial}{\partial \eta_n}|\nu_n|, \tag{3.14}$$

and as $|\nu_n| \to 1$ in $C^{1,\alpha}(U)$, one obtains $\kappa_{g_{1,n}}(\gamma) = o(||\Phi_n||^{-1/2}) = o(\mathcal{E}_n^{-1/2})$.

Then to any arc γ of the horizontal foliation of Φ_n avoiding any zeros of Φ_n, one has that is is mapped close to its geodesic in the target hyperbolic surface. The following standard calculation on the stretching shows that by normalizing the target hyperbolic manifold by the total energy, the resulting length is given by the intersection number with the measured lamination λ_1. One has

$$\begin{aligned}
l_{g_{1,n}}(\gamma) &= \int_\gamma \mathcal{H}_n^{1/2} + \mathcal{L}_n^{1/2} \, ds_{\sigma_n} \\
&= \int_\gamma \mathcal{H}_n^{1/2}(1 + |\nu_n|) \, ds_{\sigma_n} \\
&= \int_\gamma \frac{||\Phi_n||^{1/2}|\Phi_0|^{1/2}}{|\nu_n|^{1/2}}(1 + |\nu_n|) \, \frac{ds_{\sigma_n}}{\sigma_n^{1/2}} \\
&= ||\Phi_n||^{1/2} \int_\gamma \left(1 + \left(\frac{1}{|\nu_n|^{1/2}} - 1\right)\right) (2 - (1 - |\nu_n|)) \, ds_{|\Phi_0|} \\
&= 2||\Phi_n||^{1/2} l_{|\Phi_{0,n}|}(\gamma) + O(||\Phi_n||^{1/2}(1 - |\nu_n|)),
\end{aligned}$$

recalling that in order to obtain the metric σe, one has to divide both hyperbolic surfaces by twice the energy, which is approximately 4 times the L^1-norm of the Hopf differential for sufficiently large energy, independent of the Riemann surface structure (see Proposition 3.8). Meanwhile, a similar calculation shows that an arc of the vertical foliation of Φ_n avoiding the zeros of Φ_n, say α, has length in the target hyperbolic surface given by

$$\begin{aligned} l_{g_{1,n}}(\alpha) &= \int_\alpha \mathcal{H}_n^{1/2} - \mathcal{L}_n^{1/2}\ d_{s_{\sigma_n}} \\ &= \int_\alpha \mathcal{H}_n^{1/2}(1-|\nu_n|)\, ds_{\sigma_n} \\ &= \int_\alpha \frac{||\Phi_n||^{1/2}|\Phi_{0,n}|^{1/2}}{\sigma_n^{1/2}|\nu_n|^{1/2}}(1-|\nu_n|)\ ds_{\sigma_n} \\ &= ||\Phi_n||^{1/2}\int_\alpha \frac{1-|\nu_n|}{|\nu_n|^{1/2}}\ ds_{|\Phi_{0,n}|} \\ &= o(\mathcal{E}_n^{1/2}). \end{aligned}$$

Noting that a horizontal arc of Φ_n is a vertical arc of $-\Phi_n$, one sees the λ_1 and λ_2 are the horizontal and vertical foliations of Φ_∞ (the geometric limit of Φ_n, see [42]) respectively.

To get our desired harmonic map from the flat subsurface to the two trees, notice that the above estimates show that a horizontal arc of $\Phi_{0,n}$ gets mapped close to a geodesic in the target space which is a hyperbolic surface scaled by the reciprocal of total energy. As the scaled induced metric limits to the flat metric $|\Phi_\infty|$, a horizontal arc of Φ_∞ will thus be mapped by an isometry to the tree T_1 and any vertical arc collapsed, so that the limiting map in the universal cover is given by a projection onto the leaf space of the horizontal foliation of Φ_∞. The same argument holds for T_2.

□

Proposition 3.15. *For any closed curve γ on the surface S, one has the following pair of inequalities:*

$$l_{g_{1,n}}(\gamma) \leq l_{\sigma_n e_n}(\gamma)$$

$$l_{g_{2,n}}(\gamma) \leq l_{\sigma_n e_n}(\gamma)$$

Consequently if $t_n L_{\sigma_n e_n} \to \eta$ as currents, then the length spectra of $\lim_{n\to\infty} t_n L_{g_{i,n}}$ are well-defined. If the limiting currents are denoted λ_j, then

$$i(\lambda_j, \cdot) \leq i(\eta, \cdot).$$

Proof. As the minimal surface has induced metric of the form $g_{1,n} + g_{2,n}$, where the $g_{i,n}$ is a hyperbolic metric, both inequalities is immediate. The final comment follows from choosing a closed curve $\gamma = \gamma_n$ to be a $\sigma_n e_n$-geodesic and using the inequality $l_{t_n^2 g_{i,n}}([\gamma]) \leq l_{t_n^2 g_{i,n}}(\gamma)$. □

Combining Proposition 3.15 and Theorem 3.14, we obtain a necessary and sufficient condition on the pair of measured laminations λ_1 and λ_2 to determine a corresponding flat part on the mixed structure.

Corollary 3.16. *Let $\lambda_i' = \lim_{n\to\infty} X_{i,n}'/2\mathcal{E}_n$ be a pair of non-zero measured laminations on a subsurface S'. Then the pair of laminations fill if and only if the restriction of the mixed structure η to S' is flat.*

Proof. If η is flat on S', the preceding theorem shows the pair of laminations are dual and hence fill. If the pair of laminations do fill, then for any third lamination λ' one has by Propostion 3.15 that $i(\eta, \lambda') > 0$, so that it cannot be a lamination, and hence must be flat by definition of a mixed structure. □

Proposition 3.17. *On the subsurface $S'' = S \setminus S'$, the laminations λ_1 and λ_2 restrict to a pair of measured laminations which have no transverse intersection. If λ denotes the measured lamination part of the mixed structure, then $i(\lambda, \lambda_1) = i(\lambda, \lambda_2) = 0$.*

Proof. By Proposition 3.15, since $i(\lambda, \lambda) = 0$, one has that $i(\lambda_1, \lambda) = i(\lambda_2, \lambda) = 0$. Applying the inequality from Proposition 3.15 again yields $i(\lambda_1, \lambda_2) \leq i(\lambda, \lambda_2) = 0$, from which the result follows.

□

3.2.3 From geodesic currents to metric spaces

In this section, we construct noncompact metric spaces admitting a $\pi_1 S$-action by isometries.

Definition 3.18. Let X and X' be two metric spaces and let $\epsilon > 0$. Then an *ϵ-approximation* between X and X' is a relation R in $X \times X'$ that is onto, so that for every $x, y \in X$ and for every $x', y' \in X'$, the conditions xRx' and yRy' imply $|d_X(x,y) - d_{X'}(x', y')| < \epsilon$.

Definition 3.19. Let X_n be a sequence of metric spaces, each admitting an isometric action by a group Γ and a supposed limiting metric space X_∞ also admitting an isometric action by the same group Γ. Then we say X_n converges to X_∞ in the sense of Gromov-Hausdorff-Paulin, if for every $\epsilon > 0$ and every finite set $A \subset \Gamma$, and for every compact subset $K \subset X_\infty$, then for n sufficiently large, there is a compact set

$K_n \subset X_n$ and an ϵ-approximation R_n which is A-equivariant between K_n and K in the following sense: for every $x \in K$, for every $x_n, y_n \in K_n$, and for every $\alpha \in A$, we have that the conditions $\alpha x \in K$ and $x_n R_n x$ and $y_n R_n \alpha x$ imply $d(\alpha x_n, y_n) < \epsilon$.

We construct a sequence of noncompact metric spaces X_n with an isometric action by $\Gamma = \pi_1 S$ as follows. Take the induced metric $(S, \sigma_n e_n)$ and lift the metric to the universal cover $(\widetilde{S}, \widetilde{\sigma_n e_n})$. We will deal with the case where the induced metric converges in length spectrum to a mixed structure that is not entirely laminar (this is to ensure so that we can scale our metric spaces by total energy; for the case of a mixed structure that is entirely laminar, the same discussion holds after amending the sequence of constants). The sequence of noncompact metric spaces thus will be $X_n = (\widetilde{S}, \widetilde{\sigma_n e_n}/\mathcal{E}_n)$. The following proposition is thus clear.

Proposition 3.20. *The manifold $X_n = (\widetilde{S}, \widetilde{\sigma_n e_n}/\mathcal{E}_n)$ is a noncompact metric space admitting an isometric action by the group $\Gamma = \pi_1 S$.*

Proof. As X_n itself is a noncompact Riemannian manifold with $\Gamma = \pi_1 S$ acting on it by isometries, the result follows immediately. □

Up to a subsequence, the metrics $(S, \sigma_n e_n/\mathcal{E}_n)$ will converge in length spectrum to a non-trivial mixed structure $\eta = (S', q, \lambda)$. We construct a noncompact metric space $X_\infty = X_\eta$ from the mixed structure η. Regard η as a geodesic current on $(\widetilde{S}, g)$. To any two distinct points $x, y \in \widetilde{S}$, one can form the geodesic arc α connecting the two points. Let c be the set of bi-infinite geodesics which intersect α transversely. Then the intersection number $i(\eta, \alpha)$ is given by the η-measure of c. This yields a pseudo-metric space coming from the geodesic current η. Notice it is possible for the intersection number to be zero, for instance if the geodesic arc is disjoint from the support of the current, or if it forms no nontransverse intersection with the support of η. Taking the quotient by identifying points which are distance 0 from each other, and then taking the metric completion, yields a noncompact metric space X_∞. As $\Gamma = \pi_1 S$ acted on η equivariantly, then Γ acts by isometries on X_∞. For a more detailed discussion about the construction of a metric space from the data of a geodesic current, see [7].

Remark 3.21. In the setting where η is a measured foliation, the metric space X_η is a familiar one. It is a $\mathbb{R}$-tree dual to the foliation. The space is constructed by collapsing the leaves of the foliation with the distance on the tree inherited by intersection number and then completing (see [46]). The case where η is a non-trivial mixed structure follows the same spirit of this construction. The laminar part is tree-like, formed on the universal cover by collapsing leaves of the supported lamination and then completing. On the flat part, the metric space is formed by the product of the trees dual to the vertical and horizontal lamination of a quadratic differential whose metric is the given flat metric.

The preceding discussion is summarized by the following proposition.

Proposition 3.22. *To any mixed structure η, the construction above gives a non-compact metric space X_η admitting an isometric action by $\Gamma = \pi_1 S$.*

Using the Gromov-Hausdorff topology, one has the following.

Theorem 3.23. *A subsequence of the metric spaces $(\widetilde{S}, \widetilde{\sigma_n e_n}/\mathcal{E}_n)$ converges in the* *sense of Gromov-Hausdorff to a noncompact metric space X_η coming from a mixed structure η acted upon by $\Gamma = \pi_1 S$.*

Before presenting the proof, we record one useful fact regarding convergence of maps. This follows from work of Korevaar-Schoen.

Theorem 3.24 (Korevaar-Schoen, see [35], [14])**.** *Let $\widetilde{M}$ be the universal cover of a compact Riemannian manifold, and let $u_k : \widetilde{M} \to X_k$ be a sequence of maps such that:*

a. Each X_k is an NPC space

b. The u_k's have uniform modulus of continuity: For each x, there is a monotone function $\omega(x, \cdot)$, so that $\lim_{R\to 0} \omega(x, R) = 0$ and $\max_{B(x,R)} d(u_k(x), u_k(y)) \leq \omega(x, R)$.

Then the pullback metrics d_{u_k} converge (possibly after passing to a subsequence) pointwise, locally uniformly to a pseudometric d_∞.

Proof of Theorem 3.23. Recall from Theorem 3.13, that on S' we have uniform convergence of the induced metric to the flat metric. For the complementary subsurface, recall that metric spaces were obtained as the induced metric on the minimal surface, so that the metric came from a pull-back of a harmonic map. By Proposition 3.6, the scaled metric is the pull-back metric of a harmonic map with energy at most 1. Hence by Theorem 3.24 (see Proposition 3.7 [35], or Theorem 2.2 [14]), the metrics converge uniformly. As the lifts of the induced metrics admitted an $\pi_1 S$-action by isometries, so does the limit.

□

3.2.4 Convergence of Harmonic maps

Not only do the metric spaces converge in a suitable topology, the harmonic maps do as well. As we have shown in the preceding section that the domains converge in the sense of Gromov-Hausdorff to a metric space arising from a mixed structure, and as shown in work of Wolf [63], one has that the lifts of a sequence of degenerating hyperbolic metrics, when properly scaled, subconverge in the sense of Gromov-Hausdorff to $\mathbb{R}$-trees dual to a particular measured lamination in the projective class of the associated point on the Thurston boundary. Hence we have both domain and target converging in the same topology to noncompact metric spaces with isometric actions by $\Gamma = \pi_1 S$. It is natural to expect some sort of convergence in the harmonic maps. In Wolf [63], the domain is a fixed Riemann surface, and the target is changing. Here, we have both domain and target changing (and converging). We begin by reviewing the necessary definitions.

Definition 3.25 (see also [63])**.** Let X_n, X_∞ be spaces admitting an action of a group Γ and let (Y_n, d_n) and (Y_∞, d_∞) be metric spaces admitting an isometric action of Γ. Suppose $f_n : X_n \to Y_n$ and $f_\infty : X_\infty \to Y_\infty$ are equivariant maps. Then we say that f_n converges (uniformly) to f if

(i) Both X_n and Y_n converge (uniformly) to X and Y respectively in the sense of Gromov, and

(ii) For every $\epsilon > 0$, there is an $N(\epsilon)$ so that for $n > N(\epsilon)$, the ϵ-approximations R_n, R'_n satisfies: for every $x_n R_n x$ one has $f_n(x_n) R'_n f(x)$.

We will require a notion of harmonic for maps between singular spaces. The following can be found in more detail in [17].

Definition 3.26. Let $\phi \in L^2_{loc}(X, Y)$. The approximate energy density is defined for $\epsilon > 0$ by

$$e_\epsilon(\phi)(x) = \int_{B_X(x,\epsilon)} \frac{d_Y^2(\phi(x), \phi(x'))}{\epsilon^{m+2}}\, d\mu_g(x').$$

Definition 3.27. The energy $E(\phi)$ of a map ϕ of class $L^2_{loc}(X, Y)$ is

$$E(\phi) = \sup_{f \in C_c(X,[0,1])} \left(\limsup_{\epsilon \to 0} \int_X f e_\epsilon(\phi) d\mu_g \right)$$

Definition 3.28. A harmonic map $\phi : X \to Y$ is a continuous map of class $W^{1,2}_{loc}(X, Y)$ which is bi-locally E-minimizing in the sense that X can be covered by relatively compact subdomains U for each of which there is an open set $V \supset \phi(U)$ in Y such that

$$E(\phi|_U) \leq E(\psi|_U)$$

for every continuous map $\psi \in W^{1,2}_{loc}(X, Y)$ with $\psi(U) \subset V$ and $\psi = \phi$ in $X \backslash U$.

In the setting where both singular spaces are finite metric graphs, the resulting harmonic maps are *affine maps*. Each edge of the domain graph is mapped via the constant map, or mapped linearly to the target graph. The following result of Lebeau characterizes all such harmonic maps.

Theorem 3.29 (Lebeau [39])**.** *Given two finite metric graphs G and G', every continuous map between G and G' is homotopic to a affine map which minimizes the energy within its homotopy class. Furthermore, the map is unique up to parallel transport.*

Proposition 3.30. *Suppose $L_{\sigma_n e_n/C_n}$ converges to λ, where λ is a Jenkins-Strebel lamination. Then the sequence of metric spaces $(S, \sigma_n e_n/C_n)$ converges geometrically to a finite metric graph.*

Proof. This follows immediately from Theorem 3.24 (see also Proposition 3.7 of [35]), as the induced metrics are the pullback metrics of a harmonic map from $\mathbb{H}^2$ to $\mathbb{H}^2 \times \mathbb{H}^2$, which is NPC. The assumption on the modulus of continuity follows from the bound on the total energy of the maps u_n to the rescaled target, so that total energy is at most 1. Hence, the limiting metric space is the dual graph of λ, which is a finite metric graph. □

Theorem 3.31. *Let $C_n \to \infty$, so that $L_{\sigma_n e_n/C_n} \to \eta$, where η is a mixed structure with laminar part supported on a finite collection of simple closed curves. Suppose $L_{X_{i,n}/C_n} \to \lambda_i$, where λ_i are measured laminations also supported on a finite collection of simple closed curves. Then the sequence of harmonic maps $u_{i,n} : (S, \sigma_n e_n/C_n) \to X_{i,n}/C_n$ converges to a harmonic map $u_i : X_\eta \to T_i$.*

Proof. Recall X_η is the metric completion of the metric space obtained from the geodesic current η by creating a pseudo-metric space from the intersection number with η, and then identifying points with 0 distance.

As the case where η is flat has been previously handled in Theorem 3.14, we first construct a $\pi_1 S$-equivariant map between the laminar part of X_η and T_1 (here we will consider only the case where η is a Strebel lamination). The same construction will produce a similar map to T_2. Let D be a connected fundamental domain of the laminar region of X_η, then D is a finite metric graph. We embed the graph D into the laminar region S'' of the minimal surface as follows: we map each vertex of D to its corresponding thick region on S''. The geometric convergence of the minimal surfaces to D from Proposition 3.30 allows us to determine which region of the minimal surface will converge to a given vertex. Once we have made our choice of where to send each vertex of D, if there is an edge e connecting two vertices of D, then we send the edge e to the geodesic arc connecting the two points on the minimal surface where we have mapped our two vertices. (The limiting map we will obtain later will not depend on this choice, as distances will converge uniformly.)

As we have convergence in length spectrum and as there are only finitely many edges, we can ensure that for large $n > N(\epsilon)$, the length of the image of each edge has changed by at most ϵ. We require that the embedding is proportional to arclength. Then there is a collection of continuous maps $\phi_n : D \to X_n$ with the property that given $\epsilon > 0$, there is an $N = N(\epsilon)$ so that ϕ_n is a $(1+\epsilon)$ quasi-isometry.

Likewise, as $\widetilde{X}_{1,n}/C_n$ converges geometrically to an $\mathbb{R}$-tree, a fundamental domain of $\widetilde{X}_{1,n}/C_n$ will converge geometrically to a finite graph G_1 (see for instance, [63]). Hence, there is a collection of continuous maps $\psi_n : X_{1,n}/C_n \to G_1$ with the same property as ϕ_n.

Form the composition $g_n = \psi_n \circ u_{1,n} \circ \phi_n : D \to G_1$, where $u_{1,n} : (S, \sigma_n e_n/C_n) \to X_{1,n}/C_n$ is a harmonic map with total energy at most 1. We claim this sequence of maps g_n is uniformly bounded and equicontinuous. Uniform boundedness is clear as the target graph G_1 is a finite graph. To see equicontinuous, we note that as ϕ_n and ψ_n were $(1+\epsilon)$ quasi-isometries and since there is a uniform Lipschitz constant of the maps $u_{1,n}$, as the total energy of the maps are bounded by 1, (see [34], Thm

2.4.6), then equicontinuity follows. Hence, by the Arzelà-Ascoli theorem, we have a subsequence g_k converging uniformly to a map $g : D \to G_1$.

We have that g is harmonic as map between singular spaces, for we have uniform convergence of distances (see [35]) between the approximate metric spaces coming from our scaled induced metrics and the limiting $\mathbb{R}$-tree. Hence all the quantities in the definitions of the approximate energy density and the energy converge. As there is a unique energy minimizer (up to parallel transport, by Theorem 3.29) between the limiting spaces (which are finite graphs), the map g must be this unique energy minimizer. (If g were not the energy minimizer, it would have larger energy than the unique energy minimizer, by say δ. One could then construct a map between the approximate Riemannian manifolds, which would have energy lower than the harmonic maps $u_{1,n}$, contradicting the harmonicity of $u_{1,n}$.)

From Theorem 3.14, we obtained a limiting harmonic map u' on the flat part of X_η to the tree T_1, and now we have a limiting harmonic map g from the laminar part of X_η to the tree T_1. Taking the union yields the desired $u : X_\eta \to T_1$. The same argument holds for T_2.

□

3.2.5 Cores of trees

Here we review some basics of cores of $\mathbb{R}$-trees. A more detailed overview of this material may be found in [28], [63].

For any $\mathbb{R}$-tree, a *direction* at a point $x \in T$ is a connected component of $T \backslash x$. A *quadrant* in $T_1 \times T_2$ is the product of $\delta_1 \times \delta_2$ of two directions $\delta_1 \subset T_1$ and $\delta_2 \subset T_2$. We will say that the quadrant is based at $(x_1, x_2) \in T_1 \times T_2$, where x_i is the base point for the direction δ_i.

Let T_1, T_2 be a pair of trees with a common group action by Γ. Let $x = (x_1, x_2) \in T_1 \times T_2$ be a base point.

Definition 3.32. Consider a quadrant $Q = \delta_1 \times \delta_2 \subset T_1 \times T_2$. Then Q is said to be *heavy* if there exists a sequence $\gamma_k \in \Gamma$ so that

(i) $\gamma_k \cdot x \in Q$

(ii) $d_i(\gamma_k \cdot x_i, x_i) \to \infty$ as $k \to \infty$ for $i = 1, 2$.

Otherwise we say Q is *light*.

We define the *core* of a product of trees to be the product $T_1 \times T_2$ with all light quadrants removed.

Definition 3.33 (Guirardel, [28])**.** The *core* $\mathcal{C}$ of $T_1 \times T_2$ is the subset

$$\mathcal{C} = T_1 \times T_2 \backslash \left[\bigcup_{Q \text{ light quadrant}} Q \right].$$

Proposition 3.34 ([28])**.** *Let T_1 and T_2 be dual to a pair of measured foliations λ_1 and λ_2, respectively. Consider the map $p_i : \widetilde{S} \to T_i$, which maps an element of $\widetilde{S}$ to the leaf of $\widetilde{\lambda}_i$ which contains it. Then $\mathcal{C}(T_1 \times T_2) = p_1(\widetilde{S}) \times p_2(\widetilde{S})$.*

Proof. The result will follow from the claim that any quadrant $Q = \delta_1 \times \delta_2$ in $T_1 \times T_2$ is light if and only if $p_1^{-1}(\delta_1) \cap p_2^{-1}(\delta_2) = \emptyset$. It is clear that if $p_1^{-1}(\delta_1) \cap p_2^{-1}(\delta_2) = \emptyset$, then Q is light, as for each point $x \in \widetilde{S}$, the orbit of $(p_1(x), p_2(x))$ does not intersect

Q. Conversely, if $p_1^{-1}(\delta_1)$ intersects $p_2^{-1}(\delta_2)$, then take U_{δ_i} to be an open half plane in $\widetilde{S}$ with bounded Hausdorff distance from $p_i^{-1}(\delta_i)$, where U_{δ_i} is bounded by a geodesic in $\widetilde{\lambda}_i$. Then as $p_1^{-1}(\delta_1)$ has nonempty intersection with $p_2^{-1}(\delta_2)$, then so do U_{δ_1} and U_{δ_2}. Moreover, there exists a geodesic γ intersecting the pair of geodesics bounding U_{δ_1} and U_{δ_2}. Take an element $h \in \pi_1 S$ whose axis is γ. Then h is hyperbolic in both T_1 and T_2 and h makes Q heavy. □

Remark 3.35. This characterization of the core of two trees is particularly useful in our setting where the trees come from measured laminations. The map p which sends $\mathbb{H}^2$ to the leaf space of a measured lamination is a $\pi_1 S$-equivariant harmonic map, and as a product of harmonic maps is harmonic, we see that the core is the image of the $\pi_1 S$-equivariant harmonic map $(p_1 \times p_2) : \mathbb{H} \to T_1 \times T_2$.

In the setting where where T_1 and T_2 arise from two transverse measured foliations λ_1 and λ_2, then $\mathcal{C}(T_1 \times T_2)/\pi_1 S$ is isometric to S endowed with the unique singular Euclidean metric whose vertical and horizontal foliations are λ_1 and λ_2.

We present our next main result concerning the relation between the mixed structures we obtain as limits of the induced metrics and the limits of the corresponding graphs of the minimal langrangians.

Theorem 3.36. *Suppose $C_n \to \infty$, so that $L_{\sigma_n e_n/C_n} \to \eta$ and $X_{1,n}/C_n \to T_1$ and $X_{2,n}/C_n \to T_2$. Then the metric space X_η is isometric to the core of the pair of trees (T_1, T_2). Consequently, the minimal lagrangians $\widetilde{\Sigma_n}/C_n \subset \mathbb{H}^2/C_n \times \mathbb{H}^2/C_n$ converge geometrically to the core $\mathcal{C}(T_1 \times T_2) \subset T_1 \times T_2$.*

Proof. Define the auxiliary map $\Psi : \mathbb{P}(\mathcal{ML} \times \mathcal{ML}) \to \mathrm{PMix}(S)$ by

$$\Psi([\lambda_1, \lambda_2]) = \lim_{n\to\infty} [L_{\sigma_n e_n}],$$

where $\Sigma_n \subset X_{1,n} \times X_{2,n}$ is the minimal lagrangian with induced metric $2\sigma_n e_n$ and $(X_{1,n}, X_{2,n})$ converge projectively to $[(\lambda_1, \lambda_2)]$. We claim the map is well-defined.

Choose $[(\lambda_1, \lambda_2)] \in \mathbb{P}(\mathcal{ML} \times \mathcal{ML})$ and a representative $(\lambda_1, \lambda_2) \in [(\lambda_1, \lambda_2)]$. Then if both $(X_{1,n}/k_n, X_{2,n}/k_n)$ and $(Y_{1,n}/d_n, Y_{2,n}/d_n)$ converge in length spectrum to (λ_1, λ_2), then for large enough n, we will have that $X_{1,n}/k_n$ will be close to $Y_{1,n}/d_n$ as negatively curved Riemannian surfaces (and likewise for $X_{2,n}/k_n$ and $Y_{2,n}/d_n$) by [47]. Hence the induced metrics on the respective pairs of minimal langrangians will have close length spectra, so that Ψ is well-defined.

To see that Ψ is continuous, observe that the induced metric on the minimal surface varies continuously as a map defined on $\mathcal{T}(S) \times \mathcal{T}(S)$, and as the length spectrum of the induced metric varies continuously as one takes a sequence of hyperbolic surfaces $(X_{1,n}, X_{2,n}) \to [(\lambda_1, \lambda_2)] \in \mathbb{P}(\mathcal{ML} \times \mathcal{ML})$, one finds the space of mixed structures varies continuously on $\mathbb{P}(\mathcal{ML} \times \mathcal{ML})$ by a diagonal argument.

But we now have a harmonic map from X_η to $T_1 \times T_2$ defined as follows. From Theorem 3.14, the harmonic map on the flat part is given by projection to its vertical and horizontal lamination. By Theorem 3.31, the harmonic map from the laminar part is given by an affine map, when both trees come from Jenkins-Strebel differentials. We will focus on trees dual to Jenkins-Strebel differentials, as in each fundamental domain, their dual graphs are finite metric graphs.

On the region S'' of the minimal surface Σ_n/C_n, which is converging to a measured lamination $\lambda = a_1\gamma_1 + ... + a_k\gamma_k$, we may do a ϵ-thick-thin decomposition as follows. For each core geodesic in the homotopy class of γ_i, construct a cylindrical region of height $a_i - \epsilon$ with core curve given by the geodesic. These thick regions will converge to the edges of the limiting graph. The complement will be called the thin regions, which will converge to the vertices of the limiting graph. A similar decomposition may be constructed on the surfaces $X_{i,n}/C_n$. Observe that the thick regions are all cylinders and the thin regions are never homeomorphic to a cylinder.

We now claim that the limiting harmonic map from X_η to T_i will send a vertex of X_η to a vertex in T_i. If a vertex of X_η is mapped distance $\delta > 0$ away from the nearest vertex, then for sufficiently large n, the harmonic maps from Σ_n/C_n to the scaled hyperbolic surfaces $X_{i,n}/C_n$ will map at least one point p_1 in a thin neighborhood to a point in the cylindrical region which is at least distance $\delta/2$ from the boundary of the cylindrical region. The harmonic map is a diffeomorphism, so the thin region, which is never cylindrical, cannot map the thin region completely into a thick region of the scaled hyperbolic surface, so that there is at least a second point p_2 of the thin region of Σ_n/C_n mapped outside the thick region of $X_{i,n}/C_n$.

The thin regions have both diameters and systoles tending towards zero, so the distance between p_1 and p_2 tends towards zero. However, their images, by assumption, are distance at least $\delta/2$ apart. But the harmonic map is a projection, which is distance decreasing, leading to a contradiction. Hence any vertex of X_η must be mapped to a vertex of T_i.

Hence by Theorem 3.29, the map is an affine map which maps vertices to the corresponding vertices.

But the induced metric on the graph of an affine map between the two metric graphs yields the product metric for the core of the two trees (see Proposition 3.34 and the remark which follows). The equality of the mixed structure and the core of the trees then holds for pairs of $\mathbb{R}$-trees dual to a pair of Jenkins-Strebel foliations, which is a dense set in $\mathbb{P}(\mathcal{ML}\times\mathcal{ML})$, and both quantities vary continuous for $\mathbb{P}(\mathcal{ML}\times\mathcal{ML})$, thus the theorem follows.

□

Remark 3.37. In fact, by Theorems 3.14 and 3.31, the sequence of ρ-equivariant harmonic maps from $\mathbb{H}^2$ to $\mathbb{H}^2 \times \mathbb{H}^2$ converges projectively to a harmonic map from $\mathbb{H}^2$ to the product of $\mathbb{R}$-trees, whose image is the core of the trees.

3.3 Compactification of the space of maximal representations to $\mathbf{PSL}(2,\mathbb{R}) \times \mathbf{PSL}(2,\mathbb{R})$

In this section, we provide an application of our work to compactifying the maximal component of the representation variety $\chi(\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}))$. The theory of maximal representations is defined for general Hermitian Lie groups G and is considerably more straightforward to define in our specific setting of $G = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. Nevertheless, we will define a maximal representation in the general setting before providing a straightforward characterization in our setting.

Let G be a *Hermitian* Lie group, that is a noncompact simple Lie group whose symmetric space G/K is a Kähler manifold. In particular, there is a G-invariant two-form ω on G/K. Let S be a closed, orientable, smooth surface of genus $g \geq 2$. Then given a representation $\rho : \pi_1 S \to G$, one can choose any ρ-equivariant map $\tilde{f} : \widetilde{S} \to G/K$ and define the *Toledo invariant* to be

$$T(\rho) := \frac{1}{2\pi} \int_S \tilde{f}^* \omega.$$

The Toledo invariant will be well-defined for each such representation as the number obtained will not depend on the choice of $\tilde{f}$ chosen above. A well-known *Milnor-Wood* type inequality holds for the Toledo invariant,

$$|T(\rho)| \leq |\chi(s)| \cdot \mathrm{rank}(G/K).$$

Representations whose Toledo invariant attains the upperbound are known as *maximal representations*. We now restrict our attention specifically to the group $G =$ PSL(2, $\mathbb{R}$)$\times$ PSL(2, $\mathbb{R}$), whose associated symmetric space is $\mathbb{H}^2 \times \mathbb{H}^2$.

To each representation to the group $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, one obtains a pair of representations to the group $\mathrm{PSL}(2,\mathbb{R})$. By work of Goldman [26], the Euler number of representations to $\mathrm{PSL}(2,\mathbb{R})$ characterizes the connected components of the representation variety. The maximal representations are precisely those whose projections live in the Hitchin component of $\mathrm{PSL}(2,\mathbb{R})$ representations, that is, those representations that are both discrete and faithful. Hence, such a representation yields a pair of points in Teichmüller space and an associated minimal surface. We may parameterize such representations by the induced metric on the minimal surface Σ, as well as the $\rho = (\rho_1, \rho_2)$-equivariant harmonic map from $\mathbb{H}^2$ to $\mathbb{H}^2 \times \mathbb{H}^2$, given by the graph of the minimal langrangian from Theorem 2.1. As a final consequence of our study of these minimal langrangians, we obtain a compactifcation of the the maximal component of surface group representations to $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

Theorem 3.38. *Let S be a closed surface of genus $g > 1$. The space of maximal representations of $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ embeds into the space of $\pi_1 S$-equivariant harmonic maps from $\mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2$, whose graphs are minimal lagrangians. The Gromov-Hausdorff boundary of these maps is given by harmonic maps from $\mathbb{H}^2$ to $T_1 \times T_2$, where T_1 and T_2 are a pair of $\mathbb{R}$-trees coming from a projective pair of measured foliations, with image given by the core of the trees.*

Proof. To any maximal representation $\rho = (\rho_1, \rho_2)$, we may look at the two closed hyperbolic surfaces given by $X_1 = \mathbb{H}^2 \backslash \rho_1$ and $X_2 = \mathbb{H}^2 \backslash \rho_2$. This gives a clear homeomorphism between the maximal component and two copies $\mathrm{Teich}(S) \times \mathrm{Teich}(S)$ of Teichmüller space. By Theorem 2.1, we obtain a minimal lagrangian ϕ between X_1 and X_2 which respects the marking. If Σ denotes the conformal structure of the graph of ϕ, then the inclusion map $i : \Sigma \to X_1 \times X_2$ is a conformal map, which lifts to the desired ρ-equivariant map from $\mathbb{H}^2$ to $\mathbb{H}^2 \times \mathbb{H}^2$. The correspondence which associates the representation ρ to this map is continuous and is injective as distinct representations have distinct minimal lagrangians, hence yielding our desired embedding.

If ρ_n is a sequence of representations leaving all compact sets, then either $g_{1,n}$ or $g_{2,n}$ (or both) leaves all compact sets in Teichmüller space (recall $(S, g_i) = \mathbb{H}^2/\rho_i$). By Theorem 3.5 (up to a subsequence) the sequence of induced metrics on the graphs converge projectively to a mixed structure. Let c_n be the sequence of constants for which we divide the induced metric to ensure length spectrum convergence to a non-zero mixed structure with self-intersection 1 or a measured lamination. If we scale the target by the same sequence of constants, then the total energy of the sequence of harmonic maps is now uniformly bounded, so that by Theorem 3.24, the maps converge to a map from $\mathbb{H}^2$ to $T_1 \times T_2$, where T_i is the $\mathbb{R}$-tree associated to the limit $\widetilde{X}_{i,n}/c_n$ (notice that T_i may be a single point). By Theorem 3.36, the image is given by the core of the trees, which suffices for the proof.

□

3.4 Applications to maximal surfaces in AdS^3

In this section, we prove the required analogues of the minimal lagrangian setting to show a similar result for limits of maximal surfaces in GHMC AdS^3 manifolds.

3.4.1 Anti-de Sitter space

We are primarily concerned with the anti-de Sitter space of signature $(2,1)$, which is given by the quasi-sphere $x_1^2 + x_2^2 - x_3^2 - x_4^2 = -1$ inside $\mathbb{R}^{(2,2)}$ with the metric $ds^2 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$. More precisely,

$$\widehat{AdS^3} = \{x \in \mathbb{R}^{(2,2)} : \langle x, x \rangle = -1\}.$$

As the manifold is pseudo-Riemannian, tangent vectors $v \in T\widehat{AdS^3}$ come in one of the following three types:

Timelike if $\langle v, v \rangle < 0$

Lightlike if $\langle v, v\rangle = 0$

Spacelike if $\langle v, v\rangle > 0$.

The anti-de Sitter space AdS^3 is given by the projectivization of $\widehat{AdS^3}$, its double cover. The isometry group of AdS^3 is $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$.

A smooth surface $S \hookrightarrow \mathrm{AdS}^3$ is said to be *spacelike* if the restriction to S of the metric on AdS^3 is a Riemannian metric. This is equivalent to the condition that every tangent vector $v \in TS$ is spacelike.

Consider the Levi-Civita connections on S and AdS^3 given by ∇ and ∇^S, respectively. For a unit normal field N on S, the second fundamental form is given by

$$\nabla_{\tilde{v}}\tilde{w} = \nabla^S_v w + \mathrm{II}(v, w)N.$$

The shape operator is the $(1, 1)$ tensor given by $B(v) = \nabla_v N$. It satisfies the property $\mathrm{II}(v, w) = \langle B(v), w\rangle$. The maximal surfaces then are governed by the condition that $\mathrm{tr}B$=0.

An AdS^3 manifold is a Lorentzian manifold locally isometric to AdS^3. Among these manifolds, we restrict our attention to those which are"globally hyperbolic maximal compact", henceforth written as "GHMC". These manifolds are defined by those satisfying the following three properties:

1. they contain a closed orientable space-like surface S
2. each complete time-like geodesic intersects S precisely once
3. maximal with respect to isometric embeddings.

It follows that GHMC AdS^3 manifolds must be homeomorphic to $S \times \mathbb{R}$. Mess [43] showed that the genus of S must be at least 2 and that GHMC structures are parametrized by two copies of Teichmüller space. Barbot, Béguin and Zeghib [2] showed that for each such GHMC manifold, there exists a unique embedded spacelike maximal surface Σ. In fact, there is a parametrization of all such GHMC manifolds by the unique embedded maximal surface it contains along with its second fundamental form.

Theorem 3.39 (Krasnov-Schlenker [36]). *Let M be a GHMC AdS^3-manifold and let Σ be its unique embedded spacelike maximal surface. The second fundamental form of Σ is given by the real part of a holomorphic quadratic differential on the underlying complex structure of the maximal surface. Furthermore, there is a homeomorphism between the space of all GHMC AdS^3-structures and the cotangent bundle of Teichmüller space, which assigns to a GHMC AdS^3-structure, the conformal class of its unique maximal surface and the holomorphic quadratic differential for which its real part is the second fundamental form.*

The induced metric of the maximal surface is given by $e^{2u}\sigma$, where σ is the hyperbolic metric and u satisfies the following PDE:

$$\Delta_\sigma u = e^{2u} - e^{-2u}|\Phi| - 1.$$

But the solution to this PDE is $u = \frac{1}{2}\log \mathcal{H}$ for which the PDE becomes the usual Bochner equation. Here $\mathcal{H}$ is the holomorphic energy density arising from harmonic maps between closed hyperbolic surfaces. Hence, the induced metric of the maximal surface is given by $\mathcal{H}\sigma$. As a corollary of our main result, we will describe the limiting length spectrum of any sequence of induced metrics of the maximal surface.

3.4.2 Relation to minimal lagrangians

In the previous section, it was mentioned that maximal surfaces enjoy the same parameterization as the minimal lagrangians in $\mathbb{H}^2 \times \mathbb{H}^2$, namely via the bundle of holomorphic quadratic differentials over Teichmüller space. There is however, an explicit way to construct the associated minimal lagrangian started from the unique embedded spacelike maximal surface in a GHMC AdS^3 manifold. The image of the maximal surface under the Gauss map is the minimal lagrangian in the bidsic (see [36]).

3.4.3 Length spectrum compactification of maximal surfaces

Recall that to each GHMC AdS^3 manifold, there is a unique embedded spacelike maximal surface with induced metric $\mathcal{H}\sigma$, where $\mathcal{H}$ is the holomorphic energy density coming from a harmonic map $u : (S, \sigma) \to (S, \rho)$ between hyperbolic surfaces. In this section, we first prove a few basic properties concerning $\mathcal{H}$. This is an analogue of the analysis done earlier, where we observed the inability of e, the energy density, to scale linearly. Naturally, the arguments are simpler in nature as the Bochner equations already involve $\mathcal{H}$, whereas this was not the case with the energy density e.

Proposition 3.40. *On a fixed hyperbolic surface (S, σ) one has $\mathcal{H}_1 = \mathcal{H}_2$ if and only* *if* $e_1 = e_2$.

Proof. Note that if $e_1 = e_2$ then $|\Phi_1| = |\Phi_2|$ by Lemma 2.8. From $|\Phi_1| = |\Phi_2|$, one has by some basic algebra $\mathcal{L}_2 = \frac{\mathcal{H}_1 \mathcal{L}_1}{\mathcal{H}_2}$. From the Bochner formula, one has

$$\begin{aligned} \Delta \log \mathcal{H} &= 2\mathcal{H} - 2\mathcal{L} - 2 \\ \frac{1}{2}\Delta \log \frac{\mathcal{H}_1}{\mathcal{H}_2} &= (\mathcal{H}_1 - \mathcal{H}_2) - (\mathcal{L}_1 - \mathcal{L}_2) \\ &= (\mathcal{H}_1 - \mathcal{H}_2) - \mathcal{L}_1(1 - \frac{\mathcal{H}_1}{\mathcal{H}_2}). \end{aligned}$$

At a point $p \in S$ for which the quotient $\mathcal{H}_1/\mathcal{H}_2$ achieves its maximum (which without loss of generality we may assume to be greater than 1, or else as before we may reindex), the left hand side of the preceding calculation must be non-positive, but the right hand side is positive, hence $\mathcal{H}_1 = \mathcal{H}_2$ everywhere. □

Proposition 3.41. *On a fixed hyperbolic surface* (S, σ) *if* $\mathcal{H}_1 = c\mathcal{H}_2$ *then* $c = 1$.

Proof. Without loss of generality, take $c > 1$ or we we may reindex to ensure this is the case. Once again by the Bochner formula,

$$\begin{aligned} \Delta \log \frac{\mathcal{H}_1}{\mathcal{H}_2} &= 2(\mathcal{H}_1 - \mathcal{H}_2) - 2(\mathcal{L}_1 - \mathcal{L}_2) \\ 0 = \Delta \log c &= 2(c\mathcal{H}_2 - \mathcal{H}_2) - 2(\mathcal{L}_1 - \mathcal{L}_2) \\ &= 2\mathcal{H}_2(c-1) - 2(\mathcal{L}_1 - \mathcal{L}_2) \end{aligned}$$

Hence, everywhere one has

$$\mathcal{L}_1 - \mathcal{L}_2 = \mathcal{H}_2(c-1) > 0.$$

But $\mathcal{L}_1$ vanishes at the zeros of the quadratic differential Φ_1, a contradiction. Hence $c = 1$. □

Proposition 3.42. *Let* $H = \int \mathcal{H}\, dA(\sigma)$. *Then* $\mathcal{E} = 2H + 4\pi\chi$. *Consequently if* $\mathcal{E}_n \to \infty$, *then* $\lim_{n\to\infty} \mathcal{E}_n/H_n = 2$.

Proof. As $\mathcal{J} = \mathcal{H} - \mathcal{L}$ and $\int \mathcal{J}\sigma dz d\overline{z} = -2\pi\chi$, one has

$$\int \mathcal{H}\sigma dz d\overline{z} + 2\pi\chi = \int \mathcal{L}\sigma dz d\overline{z}.$$

Adding the terms yields

$$\mathcal{E} = \int (\mathcal{H} + \mathcal{L})\sigma dz d\overline{z} = 2\int \mathcal{H}\sigma dz d\overline{z} + 4\pi\chi = 2H + 4\pi\chi.$$

□

Recall from Section 2.6, the existence and uniqueness of a spacelike, embedded maximal surface in any GHMC AdS^3 manifold.

Proposition 3.43 (Lemma 3.6 [36]). *The induced metric on the maximal surface is of the form* $\mathcal{H}\sigma$.

Proposition 3.44. *The induced metric on the maximal surface has strictly negative curvature.*

Proof. The formula for curvature is given by

$$\begin{aligned} K_{\mathcal{H}\sigma} &= -\frac{1}{2\mathcal{H}\sigma}\Delta \log \mathcal{H}\sigma \\ &= -\frac{1}{2}\frac{1}{\mathcal{H}}\left(\frac{\Delta \log \mathcal{H}}{\sigma} + \frac{\Delta \log \sigma}{\sigma}\right) \\ &= \frac{-\mathcal{J}}{\mathcal{H}} \end{aligned}$$

where the last step comes from the Bochner equation and the curvature of the hyperbolic metric. □

Theorem 3.45. *There exists an embedding of the space of maximal surfaces into the space of projectivized currents.*

Proof. As the induced metrics on the maximal surfaces are negatively curved, they may be realized as geodesic currents. By Proposition 3.41, the projectivization remains injective. □

Theorem 3.46. *The closure of the space of induced metrics on the maximal surfaces is given by the space of flat metrics arising from unit norm holomorphic quadratic differentials and projectivized mixed structures.*

Proof. To any induced metric $\mathcal{H}\sigma$ on the maximal surface, there is a unique singular quadratic differential metric $|\Phi|$ associated to it. Some algebra shows that

$$\mathcal{H}\sigma = \frac{|\Phi|}{|\nu|} \geq |\Phi|,$$

which for high energy, Proposition 3.42 tells us H approximates the L^1-norm of the quadratic differential, so that if the sequence of unit-norm quadratic differentials converges to measured lamination, then so does the projective current associated to the induced metric on the maximal surface. Hence, we assume the sequence of unit-norm quadratic differential metrics converges to a mixed structure. On the flat part of the mixed structure, we know that up to a subsequence the Beltrami differentials converges uniformly to 1 outside of a small region about the zeros of the differential and a cylindrical neighborhood of the boundary curves. But then we know that on this subsurface the maximal surface metric will converge to $|\Phi_\infty|$ in terms of its length spectrum. As the total area of the mixed structure is 1 and we have normalized the maximal surface metric by the total holomorphic energy, on the complement, the area of the metric tends to 0, so that the restriction of the limiting current is a measured lamination.

□

We observe there is a rather interesting trichotomy at play here. For high energy, on the subsurface S', if the quadratic differentials converge to $|\Phi_\infty|$ then so do the associated sequence of minimal surface metrics and the sequence of maximal surface metrics.

Chapter 4

Conclusion

4.1 Future direction

This thesis is the start of a program to compactify Hitchin components from the perspective of equivariant minimal surfaces inside a symmetric space. The existence and uniqueness of such a equivariant minimal surface is completely resolved in the rank 2 setting, namely for the groups $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}), \mathrm{SL}(3,\mathbb{R}), \mathrm{Sp}(4,\mathbb{R})$ and G_2. This thesis has completely classified boundary limits of Hitchin $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ representations, by studying both the metric structure and the geometry of the minimal surfaces. In the setting of $\mathrm{SL}(3,\mathbb{R})$, the author and Tamburelli [49] have shown the boundary limits of a sequence of induced metrics of the minimal surfaces coming from Hitchin $\mathrm{SL}(3,\mathbb{R})$ representations, when viewed as projectivized geodesic currents, are mixed structures, where the flat metric comes from a holomorphic cubic differential instead of a quadratic one. This result is achieved by studying Blaschke metrics on equivariant affine spheres (see [40], [41], [49]). Forthcoming work by the author and Tamburelli will show a similar result for Hitchin $\mathrm{Sp}(4,\mathbb{R})$ representations by way of studying maximal surfaces in the pseudo-hyperbolic space $\mathbb{H}^{2,2}$, which is the space of negative definite lines in $\mathbb{R}^{2,3}$ (see [4], [11]). One possible avenue in this program is to classify the limits of the induced metrics on the minimal surfaces coming from Hitchin G_2 representations, thereby completing the list of rank 2 groups.

Little is known about existence and uniqueness of equivariant minimal surfaces in the rank 3 case. The product structure of the semi-simple group $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ suggests that this group may be the most tractable of rank 3 ones, though even in this setting, uniqueness is unknown. If stability of the graph minimal surfaces in $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$ is known, it would be curious to classify the geometric limits and see if the boundary objects are cores given by a product of three $\mathbb{R}$-trees.

In the setting of $\mathrm{Sp}(4,\mathbb{R})$, the representation variety has other distinguished connected components besides the Hitchin components. There are maximal components, which are further generalizations of Hitchin components. Equivariant minimal surfaces in the symmetric space $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$, coming from maximal $\mathrm{Sp}(4,\mathbb{R})$ representations, have been shown to exist and be unique by recent work of Collier, Tholozan and Toullise [11]. It is however, unknown if all these minimal surfaces have strictly negative sectional curvature. Ascertaining the curvature properties, or perhaps even asymptotics of the curvature would greatly aid in applying our program of compactifying distinguished connected components by using geodesic currents (see recent work of Dai-Li [13], [12]). Recall that a sufficient condition to realize a Riemannian metric on a surface by a geodesic current is that the sectional curvature is strictly negative, so proving that the equivariant minimal surfaces in $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ have strictly negative curvature would allow one to realize the induced metrics on the minimal surfaces as geodesic currents.

It would be quite interesting to see to what extent Hitchiness or maximality of the representation ρ ensures uniqueness of the ρ-equivariant minimal surface inside the symmetric space. Existence of the ρ-equivariant minimal surface uses the fact that Hitchin representations are quasi-isometric embeddings. But this is not enough to guarantee uniqueness; in the quasi-fuchsian case, it has been shown by Huang-Wang [33] that one can have arbitrarily many such minimal surfaces in quasi-fuchsian hyperbolic 3-manifolds.

Project 1. *Prove the uniqueness of ρ-equivariant minimal surface in $\mathbb{H}^2 \times \mathbb{H}^2$, where ρ is in a non-Hitchin component of the representation variety $\chi(\mathrm{PSL}_2\mathbb{R}\times \mathrm{PSL}_2\mathbb{R})$. Then study the limits of their metric structures and global geometry as one degenerates the representation.*

Foundational work of Goldman [26] handles the characterization of all the connected components of $\chi(\mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R})$ in terms of the Euler number. As an easy way to obtain non-Hitchin representations in $\chi(\mathrm{PSL}_2\mathbb{R}\times \mathrm{PSL}_2\mathbb{R})$, we consider the setting where one of the representations to $\mathrm{PSL}_2\mathbb{R}$ is Fuchsian and the other is not. A particular class of non-Fuchsian representations arise in the setting of handle-crushing harmonic maps from a surface of higher genus to a lower genus. Understanding the geometry of these maps would allow us to understand the ρ-equivariant minimal surfaces, when the non-Fuchsian representation comes from these handle-crushing maps. Naturally, the harmonic maps community would be interested in having a complete picture of the geometry of these maps. There has been work in this direction by Andy Huang in his thesis [32], but little else is known.

Our work in proving some new results concerning harmonic maps between closed surfaces of the same genus, provides an initial framework to delve into the new setting where the genus of the target surface is now smaller. The first step would be derive the relevant identities and equations (such as statements concerning the energy densities of these harmonic maps) governing the phenomenon when the harmonic map is no longer a diffeomorphism.

Perhaps the first example to consider is one where the domain surface can cover the target surface. Let S be a closed surface of genus g and S' be a surface of genus $g' = N(g-1)+1$ for $N > 1$ and $g \geq 2$. Then S' is a N-sheeted cover of S, so that there is a covering map $\pi : S' \to S$. Fix an auxiliary complex structure J on S' and a hyperbolic metric h on S. Then by Eells-Sampson [18], Al'ber [1] and Hartman [29], there is a unique harmonic map $f : (S', J) \to (S, h)$ in the homotopy class of $\pi : S' \to S$. Replacing (S, h) with (S, f^*h) ensures this unique harmonic map is actually $\pi : S' \to S$. The Hopf differential of the map is given by the $(2,0)$-part of the pull-back metric under π, and is a holomorphic quadratic differential on (S, J). Then to each point $[h]$ in $\mathrm{Teich}(S)$, one can choose the unique representative $h \in [h]$ to ensure $\pi : (S', J) \to (S, h)$ is a harmonic map. This yields a well-defined map $\Phi : \mathrm{Teich}(S) \to \mathrm{QD}((S', J))$ from the Teichmüller space of S to the vector space of holomorphic quadratic differentials on (S', J). When $g = g'$, Sampson [52] proved the map Ψ is continuous and injective, and Wolf [61] proved the map is a homeomorphism.

Naturally in this setting, we have no hope of the map being a homeomorphism as the dimension of $\mathrm{Teich}(S)$ is $6g-6$, whereas the dimension of $\mathrm{QD}((S', J))$ is $6g'-6 = 6N(g-1)$, which is bigger. However, we can try to determine if we can develop a Teichmüller theory of harmonic maps in the setting of mapping a higher genus surface to a lower one.

Question 1. *Is the map Φ injective? Is its image a vector subspace of $QD((S', J))$?*

The original proof by Schoen on the existence of a ρ-equivariant minimal surface in $\mathbb{H}^2 \times \mathbb{H}^2$, where ρ is Hitchin, uses the properness of an energy functional on Teichmüller space to force existence of a minimum. A result of Micallef-Wolfson [44] then implies there is a strict minimum for the area functional, and as energy dominates area, the resulting conformal map is unique.

There have been numerous other proofs since, each employing different techniques. Wan [58] has a direct proof calculating the first and second variations of the area functional. Wang [59] has a proof using mean curvature flow. An answer in the affirmative to Question 1 would provide a starting point to see if we can prove uniqueness of the ρ-equivariant minimal surface by adapting one of these perspectives to our setting.

If uniqueness of the ρ-equivariant minimal surface holds in this setting (one representation is Fuchsian, the other is not), we would like to study the induced metric on the minimal surface and understand the limiting metric structure. We would use the tools we have developed in this thesis using both techniques from harmonic maps and geodesic currents to extend the program of length spectrum compactification to the setting of a non-Hitchin component of the representation variety $\chi(\mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R})$. The new compactified component would also be a subspace of the space of projectivized currents, so it is natural to ask if this component shares any boundary objects with our compactification of the maximal component.

Project 2. *Determine to what extent length spectra distinguishes connected components of* $\chi(\mathrm{PSL}_2\mathbb{R}\times \mathrm{PSL}_2\mathbb{R})$.

There are more algebraic constructions utilizing tools from logic, such as ultrafilters, to construct compactifications of the various connected components of $\chi(\mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R})$. While it is sometimes difficult to describe explicitly what the boundary objects are, compactifications constructed in this school have the feature that they can separate the various connected components. This is often one of the reasons given to prefer the more abstract compactification of various components of the representation variety as opposed to using length spectra. Our proposed project would determine to what extent the boundaries of the different connected components coincide. Work in this direction has been done by Wolff in the setting $G = \mathrm{PSL}_2\mathbb{R}$ (see [64]).

Our work in three separate rank 2 settings (see [48], [49], [50]) suggests that flat metrics arising from holomorphic n-differentials and their mixed structures will play an integral role in studying limits of induced metrics on the minimal surface in the symmetric space of G. To this end, a required tool would be to show the geometric limits of holomorphic n-differentials are meromorphic n-differentials on lower complexity Riemann surfaces, extending the result of McMullen [42] and Ouyang-Tamburelli [49], [50] to all n.

Project 3. *Classify the geometric limits of holomorphic* n*-differentials, for* $n > 3$.

Finally, in a separate direction from the higher rank Teichmüller theory, there is the question about limits flat metrics coming from holomorphic n-differentials. To any holomorphic n-differential say $\phi = f(z)\,dz^n$, one may form a singular Riemannian metric $|\phi|^{2/n} = |f(z)|^{2/n}|dz|^2$. This will be a flat singular metric away from the zeros of ϕ, and will have a conical singularity with angle $2\pi + \frac{2\pi k}{n}$ at a zero of order k.

It would be of independent interest to the flat metrics community to ascertain the limits of all flat metrics arising from holomorphic n-differentials. Previous work by Duchin-Leininger-Rafi [16] and Ouyang-Tamburelli [49], [50] suggest the limits of any particular stratum (fixed n) of flat metrics should be mixed structures, where the flat part is a n-differential metric, but it is unclear what all the limiting objects are when one takes a sequence of flat metrics where say the n is strictly increasing. For instance, it is possible to obtain all negatively curved Riemannian metrics by considering first a very fine triangulation, then replacing each triangle with its CAT(0) comparison Euclidean triangle to obtain a C^0-approximation of the initial negatively curved metric with a NPC Euclidean cone metric. The NPC Euclidean metric can then be approximated by a flat metric coming from a holomorphic n-differential.

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